Optimal and sub-optimal power management in broadband vibratory energy harvesters with one-directional power flow constraints

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ABSTRACT

In many applications of vibratory energy harvesting, the external disturbance are most appropriately modeled as broadband stochastic processes. Optimization of power generation from such disturbances is a feedback control problem, and solvable via a LQG control theory. However, attainment of this performance requires the power conversion system which interfaces the transducers with energy storage to be capable of bi-directional power flow, and there are many applications where this is infeasible. One of the most common approaches to power extraction with one-directional power flow constraints is to control the power conversion system to create a purely resistive input impedance, and then to optimize this effective resistance for maximal absorption. This paper examines the optimization of broadband energy harvesting controllers, subject to the constraint of one-directional power flow. We show that as with the unconstrained control problem, it can be framed as a "Quadratic-Gaussian" stochastic optimal control problem, although its solution is nonlinear and does not have a closed-form. This paper discusses the mathematics for obtaining the optimal power extraction controller for this problem, which involves the stationary solution to an associated Bellman-type partial differential equation. Because the numerical solution to this PDE is computationally prohibitive for harvester dynamics of even moderate complexity, a sub-optimal control design technique is presented, which is comparatively simple to compute and which exhibits analytically-computable lower bounds on generated power Examples focus a nondimensionalized, ideal, base-excited SDOF resonator with electromagnetic transduction.

Keywords: Energy harvesting, vibration, optimal control, mechatronics

1. INTRODUCTION

Electromechanical systems to scavenge vibratory energy have been the subject of considerable research activity over the last decade, with the focus being primarily on low-power applications requiring energy-autonomy, such as wireless sensing and embedded computing systems.[1–3] Fig. 1 shows a conceptual diagram for a single-transducer vibratory energy harvesting system. The diagram is specific to harvesters with electromagnetic coupling. The power-electronic circuitry attached to the transducer regulates the rate of power extraction, $-i(t)v(t)$, from the harvester. The circuitry has several components. From left to right, these consist of an active MOSFET bridge rectifier, a buck-boost switchmode DC/DC converter, and an energy storage device such as a supercapacitor or rechargeable battery.

This particular circuit, and variants thereof, have become the subject of considerable interest for low-power energy harvesting applications.[4–9] The reasoning for its appeal stems from a number of observations. The MOSFETs in the rectifier are gated to mimic the behavior of ideal diodes, and are used in place of passive diodes. The motivation to do this is that the conductive losses in passive diodes grow with the absolute value of the current (i.e., "$v_d|i|$" losses, where v_d is the conduction voltage of a diode), whereas actively-gated MOSFETs exhibit resistive (i.e., "i^2R") conduction losses. As such, for small-current applications, active bridges can be more efficient, even though they require a small amount of parasitic power to perform the charge/discharge gating operations. Meanwhile, the DC/DC converter is controlled via high-frequency pulse-width-modulated switching of its MOSFET. The particular regime in which this converter is controlled is called "discontinuous conduction." Upon the leading edge of each switching cycle, the MOSFET is gated on, which connects the inductor to the

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Active and Passive Smart Structures and Integrated Systems 2010, edited by Mehrdad N. Ghasemi-Nejhad, Proc. of SPIE Vol. 7643, 76430Y · © 2010 SPIE · CCC code: 0277-786X/10/$18 · doi: 10.1117/12.847900

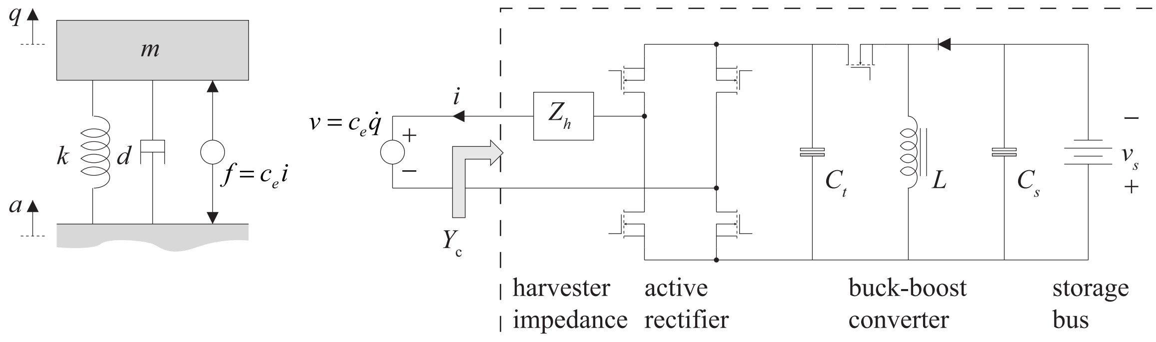


Figure 1. Energy harvesting circuit, consisting of an active bridge rectifier, and a buck/boost DC-DC converter

transducer-side capacitor C_t, and induces a current in the inductor L. After a fraction D of the total switching period, the MOSFET is gated off, which causes the inductor current to be routed to the storage bus capacitor C_s and the battery, and results in a demagnetization of the inductor current. In the discontinuous conduction regime, the inductor fully demagnetizes (i.e., its current drops to zero) before the end of the switching cycle, and remains so until the converter's MOSFET is gated on again at the leading edge of the next switching cycle.

Operation of the buck/boost converter in discontinuous conduction has the advantage that its results in an input admittance Y_c which is approximately decoupled from the behavior of the storage voltage v_s. Furthermore, if C_t is sufficiently small, and possibly with supplemental input filtering, Y_c can be made to look resistive at low frequency, with an effective value which is proportional to the inverse-square of duty cycle D; i.e.,

$$R_c = R_{c0} + R_{c1} D^{-2} \tag{1}$$

for D on the open interval from $(0, D_{max})$, where D_{max} is the duty cycle at which the converter transitions from the discontinuous to continuous conduction regime (i.e., at which the inductor no longer fully demagnetizes in the second part of each switching cycle). As such, an advantage of this approach is that one can adjust the effective resistance of the power electronics by adjusting a static parameter (i.e., D), and the resultant electromechanical dynamics of the harvester can be approximated as equivalent to those with the electronics replaced by a time-varying resistive shunt. It also means that one can optimize the power extraction for the system by optimizing the converter's admittance Y_c, over the range

$$Y_c \in [Y_c^{min}, Y_c^{max}] = \left[0, \frac{D_{max}^2}{R_{c0} D_{max}^2 + R_{c1}}\right] \tag{2}$$

This is helpful because, for a fixed value of Y_c, the harvester's dynamics are linear, and thus the dynamic response of the system is easy to analyze. The power delivered to storage is the power extracted from the transducer, minus the resistive losses in the converter. If we approximate these losses as resistive, with some resistance $R > R_{c0}$, then this power generation is

$$P_{gen}(t) = -i(t)v(t) - Ri^2(t) \tag{3}$$

The parametric optimization of Y_c, for a harvester with linear dynamics, subject to constraint (2), for maximal P_{gen} as in (3), is a quite tractable problem, both for harmonic as well as stochastic broadband disturbance characeterizations for $a(t)$.[10–12]

However, the fact that Y_c is adjustable, via changes in D, motivates questions as to whether the average value of P_{gen} might be enhanced if $Y_c(t)$ were adapted in time, in response to the dynamics of the harvester. Potential improvements in power generation with adaptive $Y_c(t)$ are the focus of this paper. It turns out that the most straght-forward manner in which to analyze this problem involves the use of optimal control theory, with $i(t)$ treated as a control input to the system, and subject to the directional power flow constraint

$$i(t)v(t) + i^2(t)/Y_c^{max} \leqslant 0 \quad , \quad \forall t \tag{4}$$

on feasibility of $i(t)$. Note that since $i(t) = -Y_c(t)v(t)$, this constraint is identical to that in (2).

For this discussion, we consider the case in which $a(t)$ is modeled as filtered noise. We will assume that $a(t)$ has a power spectral density equal to

$$\Phi_a = \left| \frac{j\omega q}{-\omega^2 + \omega_a^2 + j2\omega_a\zeta_a\omega} \right|^2 \tag{5}$$

where ω_a is the center of the passband of $a(t)$, and ζ_a determines the spread of its frequency content. The parameter q is adjusted such that irrespective of ω_a and ζ_a, $a(t)$ has a consistent standard deviation of σ_a; i.e.,

$$\sigma_a = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_a(\omega) d\omega} \tag{6}$$

This allows us to compare disurbances of varying spectral content, but equal intensity. We refer to the "narrowband limit" for the disturbance model as the case in which $\zeta_a \to 0$. Similarly, refer to "broadband limit" as the case in which $\zeta_a \to \infty$.

To investigate the use of adaptive admittance to enhance power generation, we will resort the simple example of a SDOF, stochastically-excited oscillator characterized by

$$m\ddot{r}(t) + d\dot{r}(t) + kr(t) = ma(t) + c_e i(t) \qquad v(t) = c_e \dot{r}(t) \tag{7}$$

This system corresponds to an energy harvester with electromagnetic coupling.[7,13] To make the analysis as general as possible, we first nondimensionalize this system, via the change of variables $t \leftarrow \sqrt{m/k}\ \tau$, $r(t) \leftarrow (m\sigma_a/k)\bar{r}(\tau)$, $a(t) \leftarrow \sigma_a \bar{a}(\tau)$, $i(t) \leftarrow (m\sigma_a/c_e)\bar{i}(\tau)$, and $v(t) \leftarrow (c_e \sigma_a \sqrt{m/k})\bar{v}(\tau)$. In these nondimensionalized coordinates, the system is

$$\ddot{\bar{r}}(\tau) + \beta\dot{\bar{r}}(\tau) + \bar{r}(\tau) = \bar{a}(\tau) + \bar{i}(\tau) \qquad \bar{v}(\tau) = \dot{\bar{r}}(\tau) \tag{8}$$

where $\beta = d/\sqrt{km}$ is the nondimensional viscosity of the harvester. Note that in this representation, $\sigma_{\bar{a}} = 1$. The power P_{gen} can be nondimensionalized as well, via

$$\bar{P}_{gen} = \frac{P_{gen}}{m\sigma_a^2\sqrt{m/k}} = -\bar{i}(\tau)\bar{v}(\tau) - \bar{R}\bar{i}^2(\tau) \tag{9}$$

where $\bar{R}$ is the normalized dissipation resistance, equal to $\bar{R} = (\sqrt{km}/c_e)R$. The effective admittance $Y_c(t)$ can be similarly nondimensionalized to the value $\bar{Y}_c(\tau) = (c_e/\sqrt{km})Y_c(t)$. This results in the nondimensionalized maximum admittance of $\bar{Y}_c^{max}$. The power flow constraint retains the same form in the nondimensionalized system; i.e.,

$$\bar{i}(\tau)\bar{v}(\tau) + \frac{1}{\bar{Y}_c^{max}}\bar{i}^2(\tau) \leqslant 0 \tag{10}$$

We assume that the harvester has been tuned such that its resonant frequency is in the center of the disturbance passband. This implies that the power spectrum of $\bar{a}$ can be expressed (in normalized frequency $\bar{\omega} = \omega\sqrt{m/k}$) as

$$\Phi_{\bar{a}}(\bar{\omega}) = \left| \frac{\bar{q}\bar{\omega}}{1 - \bar{\omega}^2 + 2j\zeta_a\bar{\omega}} \right|^2 \tag{11}$$

with $\bar{q}$ chosen such that $\frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{\bar{a}}(\bar{\omega})d\bar{\omega} = 1$. It turns out that the value of $\bar{q}$ which brings this about is $\bar{q} = 2\sqrt{\zeta_a}$.

As such, for the nondimensionalized SDOF system, the harvesting problem is fully parametrized by the quantities β, $\bar{R}$, $\bar{Y}_c^{max}$, and ζ_a. In the sequel, we will uniformly assume the problem to be nondimensionalized as such, and to ease the notation, do away with all overbars.

2. BROADBAND OPTIMIZATION OF STATIC ADMITTANCE

Consider that the nondimensionalized SDOF system can be modeled via a four-dimensional state vector $x = \begin{bmatrix} r & \dot{r} & a & \dot{a} \end{bmatrix}^T$, with dynamics goverend by

$$\dot{x} = Ax + Bi + Gw \qquad\qquad v = B^T x \tag{12a}$$

where w is a unit-intensity white noise process, and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -\beta & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2\zeta_a \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2\sqrt{\zeta_a} \end{bmatrix} \tag{13}$$

The dynamics of the harvester, with the electronics implementing the constant admittance Y_c, are governed by

$$\dot{x} = \left[A - BY_cB^T\right] x + Gw \tag{14}$$

In this case the stationary covariance matrix $S = \mathcal{E}xx^T$ is found via the Lyapunov equation

$$0 = \left[A - BY_cB^T\right] S + S\left[A - BY_cB^T\right]^T + GG^T \tag{15}$$

and the resultant average generated power is

$$\mathcal{E}P_{gen} = \mathcal{E}\left\{-iv - Ri^2\right\} \tag{16}$$
$$= \left(Y_c - Y_c^2 R\right) B^T S B \tag{17}$$

For a system as simple as this, with only one design parameter (i.e., Y_c), the most straight-forward way to optimize $\mathcal{E}P_{gen}$ is via a one-dimensional line search. One can, for example, employ the bisection algorithm to converge rapidly to the optimal Y_c, given β, R, and ζ_a. This is done in Fig. 2 for two values of β, over a subtomain in $\{\zeta, R\}$ space. Results are shown for both P_{gen} and Y_c.

3. OPTIMIZATION OF BROADBAND HARVESTING IN THE ABSENCE OF POWER FLOW CONSTRAINTS

Suppose that in place of the energy harvesting circuit in Fig. 1, one were to use a four-quadrant PWM-controlled H-bridge switching circuit, capable of high-precision, unconstrained tracking of a desired current i. Such a circuit has been demonstrated in Ref. 14. It would not be subject to the power flow constraint

$$iv + i^2/Y_c^{max} \leqslant 0. \tag{18}$$

Rather, the optimal current i could be controlled without constraints to optimize $\mathcal{E}P_{gen}$. In this case, it has been shown[15,16] that optimal physically-attainable value of $\mathcal{E}P_{gen}$ is achieved by relating $i(t)$ to the concurrent state $x(t)$ via the time-invariant feedback law

$$i(t) = Kx(t) \tag{19}$$

where

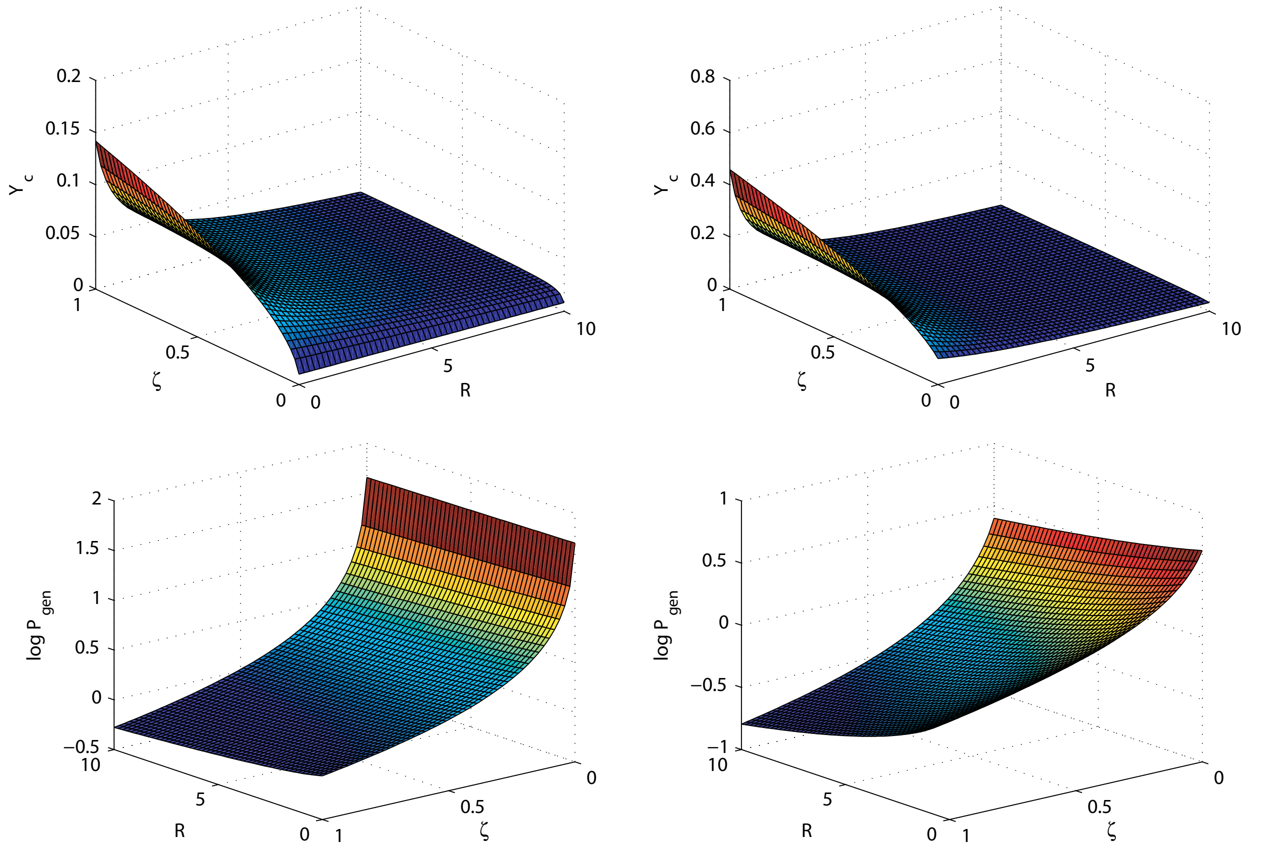
$$K = -\tfrac{1}{R}B^T\left(P + \tfrac{1}{2}I\right) \tag{20}$$

and where the matrix $P = P^T$ is the particular solution to the nonstandard algebraic Riccati equation

$$0 = A^T P + PA - \tfrac{1}{R}\left(P + \tfrac{1}{2}I\right)BB^T\left(P + \tfrac{1}{2}I\right)^T \tag{21}$$

which stabilizes the matrix $A + BK$. The resultant value of $\mathcal{E}Pgen$ is then obtained as

$$\mathcal{E}P_{gen} = -G^T P G \tag{22}$$



(c) $\beta = 0.01$ (d) $\beta = 0.1$

Figure 2. Dependency of the optimal constant Y_c and the resultant optimal average generated power $\mathcal{E}P_{gen}$, on ζ_a and R, for $\beta = 0.1$ and 0.01

Even if a harvesting circuit such as that in Fig. 1 is used to regulate power generation, the above theory is useful because one can derive an insurpassable bound on the amount of power a given hardware configuration can be expected to deliver. For the filtered-noise-excited SDOF oscillator we consider in this paper, the above equations result in some very interesting simplifications. Consider that Riccati equation (21), in this case, can be written out as

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & -\beta & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & -2\zeta_a \end{bmatrix} P + P \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -\beta & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2\zeta_a \end{bmatrix} - \frac{1}{R}\left(P + \tfrac{1}{2}I\right) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(P + \tfrac{1}{2}I\right) = 0 \qquad (23)$$

It turns out that the solution for P has the special form

$$P = \begin{bmatrix} P_{22} & 0 & P_{24} & 0 \\ 0 & P_{22} & 0 & P_{24} \\ P_{24} & 0 & P_{44} & 0 \\ 0 & P_{24} & 0 & P_{44} \end{bmatrix} \qquad (24)$$

where

$$P_{22} = -\left(\tfrac{1}{2} + \beta R\right) + \sqrt{\beta^2 R^2 + \beta R} \tag{25}$$

$$P_{24} = R\frac{-\left(R\beta + \frac{1}{2}\right) + \sqrt{\beta^2 R^2 + \beta R}}{2R\zeta_a + \sqrt{\beta^2 R^2 + \beta R}} \tag{26}$$

$$P_{44} = \frac{R}{4\zeta_a}\left(2\frac{-\left(R\beta + \frac{1}{2}\right) + \sqrt{\beta^2 R^2 + \beta R}}{2R\zeta_a + \sqrt{\beta^2 R^2 + \beta R}} - \left(\frac{-\left(R\beta + \frac{1}{2}\right) + \sqrt{\beta^2 R^2 + \beta R}}{2R\zeta_a + \sqrt{\beta^2 R^2 + \beta R}}\right)^2\right) \tag{27}$$

Thus, we have a symbolic solution for the optimal energy harvesting current as this special case, applied to (19); i.e.,

$$i(t) = \left(\frac{\beta R - \sqrt{\beta^2 R^2 + \beta R}}{R}\right) v(t) - \left(\frac{-\left(R\beta + \frac{1}{2}\right) + \sqrt{\beta^2 R^2 + \beta R}}{2R\zeta_a + \sqrt{\beta^2 R^2 + \beta R}}\right) a(t) \tag{28}$$

where we have used the fact that $v(t)$ and $a(t)$ are the second and fourth states in $x(t)$, respectively. Furthermore, we have a symbolic solution for the corresponding optimal harvested power, as

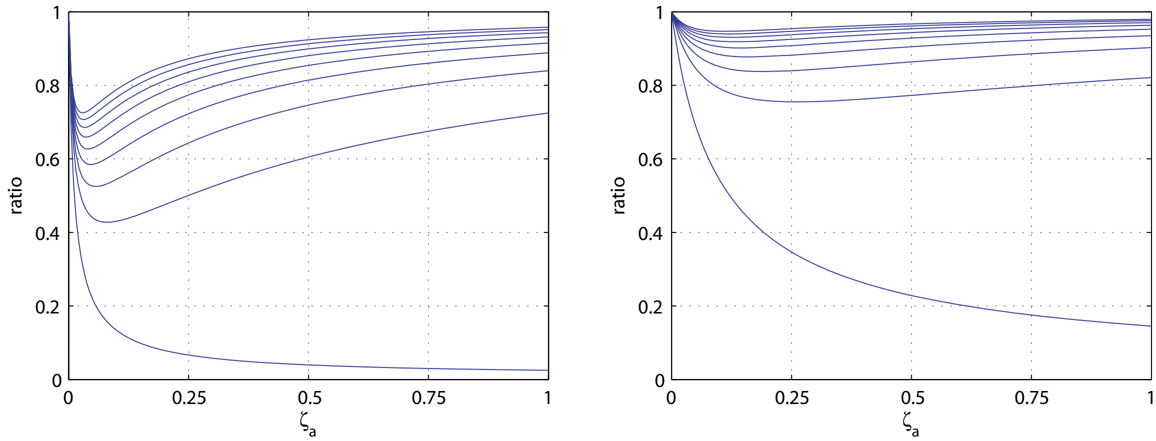
$$\mathcal{E}P_{gen} = R\left(-2\frac{-\left(R\beta + \frac{1}{2}\right) + \sqrt{\beta^2 R^2 + \beta R}}{2R\zeta_a + \sqrt{\beta^2 R^2 + \beta R}} + \left(\frac{-\left(R\beta + \frac{1}{2}\right) + \sqrt{\beta^2 R^2 + \beta R}}{2R\zeta_a + \sqrt{\beta^2 R^2 + \beta R}}\right)^2\right) \tag{29}$$

We pause for a moment to make a few brief observations regarding these results:

- It is interesting that in this particular example, the optimal manner in which to harvest power is to relate $i(t)$ only to $v(t)$ and $a(t)$, as in (28), but not to the other two states in x. This is a consequence of the very particular mathematical structure of the SDOF oscillator and noise filter we have considered, and this observation does not hold generally. For example, if a harvester input impedance exhibited significant capacitance, such as would be the case with a piezoelectric transducer, then the optimal $i(t)$ would in general only be attatinable with knowledge of the entire state vector $x(t)$. Similarly, if we had used a different disturbance filter model, the optimal $i(t)$ would require knowledge of $a(t)$, together with all its finite derivatives.

- It is also interesting to examine the symbolic dependency of the optimal feedback law in (28), on the parameters β, ζ_a, and R. Referring to the gains K_v and K_a in (28) (i.e., $i = K_v v + K_a a$), we notice that as $R \to 0$, both these gains go to infinity. It is also interesting that K_v is independent of the bandwidth of $a(t)$. Meanwhile, K_a reduces in magnitude from its value for harmonic excitation (with $\zeta_a = 0$) to the infinite-broadband case, for which $K_a \to 0$. For the narrowband case, K_a is significantly nonzero, implying that even when $a(t)$ is nearly harmonic, it is still the case that explicit knowledge of $a(t)$ may be leveraged to improve harvesting performance. However, in the limit as $\zeta_a \to 0$, both $a(t)$ and $v(t)$ become purely sinusoidal, and exactly in phase. (The phase condition is a consequence of the fact that the harvester is assumed to be tuned to the center of the passband for $a(t)$.) Thus, in this limiting case, knowledge of both $v(t)$ and $a(t)$ is redundant, as one is known to be a scaled version of the other. We can therefore conclude that in this case, the optimal $i(t)$ is in fact attained by imposing constant Y_c. This observation is harmonious with what we expect from approaching the harmonic energy harvesting problem from an impedance matching perspective.[16]

- The optimal $\mathcal{E}P_{gen}$, as expressed in (29), is always positive. Furthermore, we see that it increases monotonically as ζ_a decreases. This is to be expected, as it stands to reason that for disturbance models of equal acceleration intensity, ones with more signal strength concentrated near resonance will be easier to harvest energy from. Although it is less obvious from (29), it is also the case that $\mathcal{E}P_{gen}$ decreases monotonically as R increases. This is also to be expected, as it stands to reason that as the electronics become less efficient, the harvesting potential decreases.



(a) $\beta = 0.01$ (b) $\beta = 0.1$

Figure 3. Ratio of optimal $\mathcal{E}P_{gen}$ values for constant Y_c and fully active harvesting controllers, for R values of $0, 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2$ (from bottom to top)

The ratio of the optimal value of $\mathcal{E}P_{gen}$ for the case with constant Y_c, over that for the case with fully active control, gives us an idea of the potential for improvement in energy harvesting performance, beyond the constant-Y_c design. Clearly, this margin of improvement will depend on the system parameters; i.e., for the nondimensionalized SDOF example, β, ζ_a, and R. Fig. 3 shows these ratios for various values of β and R, and for ranges of $\zeta_a \in [0, 1]$. As can be seen from these plots, there is clearly a finite bandwidth for $a(t)$ at which explicit knowledge of $a(t)$ is most beneficial, for all cases except the asymptotic case where $R \to 0$.

4. OPTIMIZATION OF ENERGY HARVESTING WITH POWER FLOW CONSTRAINTS

We now consider the problem of optimizing $\mathcal{E}P_{gen}$ subject to constraint (18). In this section, we begin by considering the approach to solving this problem for the true optimum. This is done in Sec. 4.1. However, the methods we will ultimately consider are sub-optimal methods, which have the advantage of being much easier to deal with, mathematically. These are discussed in Sec. 4.2.

4.1 Power-flow-constrained energy harvesting as a Hamilton-Jacobi problem

Let the average power generated over a time interval $t \in [0, T]$ be denoted as

$$\tilde{P}_{gen}(T) = \frac{1}{T}\int_0^T P_{gen}(t)dt \tag{30}$$

Then our objective is to find optimal nonlinear relationship $i(t) = \phi_T(x(t), t)$ which maximizes the expectation of $\tilde{P}_{gen}(T)$. The subscript in $\phi_T(\cdot,\cdot)$ reflects the fact that this nonlinear relationship will change depending on the time duration over which energy is to be harvested. The limiting case as $T \to \infty$; i.e., $i(t) = \phi_\infty(x(t), t)$ corresponds to the case of optimal stationary energy harvesting, which is the result we ultimately want to find. For any finite T denote

$$J_T = -\mathcal{E}\bar{P}_{gen}(T) = \frac{1}{T}\mathcal{E}\left\{\int_0^T \left(x^T(t)Bi(t) + Ri^2(t)\right)dt\right\} \tag{31}$$

then equivalently, our objective is to find the relationship $i(t) = \phi_T(x(t), t)$ which minimizes J_T. In the minimization of (31), we presume the system to be subject to power flow constraint (18) at every time t.

This problem statement falls into the standard framework of stochastic Hamilton-Jacobi theory.[17] Define the function $V_T(x, t)$ as the negative of the expected energy extraction over the time interval $[t, T]$, predicated on knowledge of the state $x(t)$ at time t; i.e.,

$$V_T(x(t),t) = \mathcal{E}\left\{ \int_t^T \left(x^T(t)Bi(t) + Ri^2(t)\right) dt \;\middle|\; x(t) \right\} \tag{32}$$

Then Hamilton-Jacobi theory dictates that the determination of the optimal $\phi_T(x(t),t)$ involves the solution to the Hamilton-Jacobi equation (HJE), which is a partial differential equation in both x and t. Define the spatial gradient vector of $V_T(x(t),t)$ as

$$p_T(x(t),t) = \frac{\partial V_T}{\partial x} \tag{33}$$

and the corresponding Hessian as

$$H_T(x(t),t) = \frac{\partial^2 V_T}{\partial x \partial x^T} \tag{34}$$

Then for this problem, the HJE is

$$-\frac{\partial V_T}{\partial t} = \min_{x^T Bi + i^2/Y_c^{max} \leqslant 0} \left\{x^T Bi + Ri^2 + p_T^T(x,t)\left(Ax + Bi\right)\right\} + \tfrac{1}{2} G^T H_T(x,t) G \tag{35}$$

which must be solved for a nontrivial solution over $\{x,t\} \in \mathbb{R}^n \times [0,T]$, with the final-value condition

$$V_T(x,T) = 0 \quad , \quad \forall x \tag{36}$$

The resultant optimal feedback relationship $\phi_T(x(t),t)$ is then the minimizer of the bracketed term on the right-hand side of the HJE; i.e.,

$$\phi_T(x(t),t) = \underset{x^T Bi + i^2/Y_c^{max} \leqslant 0}{\operatorname{argmin}} \left\{x^T(t)Bi + Ri^2 + p_T^T(x(t),t)Bi\right\} \tag{37}$$

We can actually find the expression for $\phi_T(x(t),t)$ in closed-form, in terms of $x(t)$ and $p(x(t),t)$. The above is equiavelnt to

$$\phi_T(x,t) = \underset{x^T Bi + i^2/Y_c^{max} \leqslant 0}{\operatorname{argmin}} \left\{ \left(x + p_T(x,t)\right)^T Bi + Ri^2 \right\} \tag{38}$$

where we have suppressed the time-dependence of $x(t)$ for brevity. Consider the unconstrained minimum of the expression in the brackets above. It can be shown that this expression is minimized by

$$i_u(x,p_T) = -\frac{1}{2R} B^T (p_T + x) \tag{39}$$

Thus we have that $\phi_T(x,t)$ is equivalent to

$$\phi_T(x,t) = \underset{x^T Bi + i^2/Y_c^{max} \leqslant 0}{\operatorname{argmin}} \left| i - i_u(x, p_T(x,t)) \right| \tag{40}$$

Combining this expression with (39) gives $\phi_T(x,t) = i_c(x, p_T(x,t))$, where

$$i_c(x,p_T) = \underset{x^T Bi + i^2/Y_c^{max}}{\operatorname{sat}} \left\{ -\frac{1}{2R} B^T (p_T + x) \right\} \tag{41}$$

As such, the HJE can be written explicitly as

$$-\frac{\partial V_T}{\partial t} = \left(p_T + x\right)^T Bi_c(x,p) + Ri_c^2(x,p_T) + p_T^T Ax + \tfrac{1}{2} G^T H_T G \tag{42}$$

$$= -Ri_c^2(x,p_T) + p_T^T Ax + \tfrac{1}{2} G^T H_T G \tag{43}$$

with i_c found as in (41).

Solution $V_T(x,t)$ thus requires the backward integration of $V_T(x,t)$ from $t = T$, to $t = 0$, with the final-value condition $V_T(x,T) = 0$. The resultant expectation on the optimal average power generated over the interval $t \in [0,T]$, with an initial condition x_0 is then

$$\mathcal{E}\tilde{P}_{gen}(T) = -\tfrac{1}{T}V_T(x_0, 0) \tag{44}$$

The stationary case is obtained in the limit as $T \to \infty$. It the stationary case, the dependency of $\mathcal{E}\tilde{P}_{gen}(\infty)$ on the initial condition x_0 vanishes. The functions for $p_T(x,t)$ and $H_T(x,t)$ stabilize for fixed x, in reverse time, and thus as $T \to \infty$, these functions become time-invariant functions of x; i.e., $p_\infty(x)$ and $H_\infty(x)$. A corollary of this fact is that in stationarity, the optimal average power generation turns out to be

$$\mathcal{E}P_{gen} = -\tfrac{1}{2}G^T H_\infty(0)G \tag{45}$$

Moreover, the resultant controller $\phi_\infty(x)$ which achieves this power generation is the time-invariant, nonlinear relationship

$$\phi_\infty(x) = i_c(x, p_\infty(x)) \tag{46}$$

The difficulty with the use of Hamilton-Jacobi theory is that it generally requires that we numerically solve the HJE for V, or more to the point, the function $p_\infty(x)$. Even for moderate-dimensional problems, this problem can be challenging. Moreover, once $p_\infty(x)$ has been solved, it will not constitute a closed-form expression, but rather a numerical solution which must be interpolated to determine a continuous approximation. The resultant complexity associated with this method creates barriers to its practicality. Beyond the numerical issues involved in getting $p_\infty(x)$, the resultant controller would be difficult to implement without a digital signal processor. This calls into question whether the theory is in line with the low-power applications often considered in energy harvesting.

For these reasons, the remainder of the paper is devoted to illustrating a way to approximate the stationary solution to the HJE. The method is sub-optimal, but has the analytical selling point that it is guaranteed to out-perform the optimal constant Y_c value.

4.2 Sub-optimal, power-flow-constrained energy harvesting

We now discuss an analytically-tractable manner by which we can achieve *some* of the benefit of nonlinear adaptation of $Y_c(t)$, to achieve performance which is better than that with constant Y_c. Let the optimal value of Y_c, obtained via the methods discussed in Sec. 2, be denoted $Y_c^* \in (0, Y_c^{max})$. Let the corresponding value of S, resulting from (15) with $Y_c = Y_c^*$, be denoted S^*. Then we have that the performance with $Y_c(t) = Y_c^*$, $\forall t$, is $\mathcal{E}P_{gen}|_{Y_c(t)=Y_c^*} = (Y_c^* - Y_c^{*2}R)B^T S^* B$.

Now, we note an important theorem from nonlinear stochastic control, which is proved, for example, in reference 18.

THEOREM 1. *Let $i(t) = \phi(x(t))$ be any continuous nonlinear relationship. Then with this relationship imposed on a general state space system characterized by* (12)*, the average power generation is*

$$\mathcal{E}P_{gen} = (Y_c^* - Y_c^{*2}R)B^T S^* B + R\mathcal{E}\left\{(i - Kx)^2 - (-Y_c^* v - Kx)^2\right\} \tag{47}$$

where

$$K = -\tfrac{1}{R}B^T\left(P + \tfrac{1}{2}I\right) \tag{48}$$

and P is the unique solution to the Lyapunov equation

$$0 = \left[A - BY_c^* B^T\right]^T P + P\left[A - BY_c^* B^T\right] + B\left(-Y_c^* + Y_c^{*2}R\right)B^T \tag{49}$$

In the above theorem we see that the expression for $\mathcal{E}P_{gen}$ in (47) is a summation of two terms. The first of these is actually the value of $\mathcal{E}P_{gen}$ if $Y_c(t) = Y_c^*$, $\forall t$. The second term (i.e., the expectation) does not in general have a closed form. However, we know that for any x, there always exists at least one corresponding i which makes the argument in the expectation negative, because for every time, $i(t) = -Y_c^* v(t)$ is guaranteed to satisfy feasibility criterion (18).

It is therefore a provable fact that the nonlinear relationship characterized by the simple saturation

$$i(t) = \underset{x^T(t)Bi + i^2/Y_c^{max} \leqslant 0}{\operatorname{argmin}} (i - Kx(t))^2 \tag{50}$$

$$= \underset{vi + i^2/Y_c^{max} \leqslant 0}{\operatorname{sat}} \{Kx(t)\} \tag{51}$$

will guarantee to increase $\mathcal{E}P_{gen}$ beyond the case with $Y_c(t) = Y_c^*$, $\forall t$. The function $\operatorname{sat}(\cdot)$ is the saturation function, which restricts $i(t)$ to the power flow constraint. Note that this observation is actually true irrespective of whether the harvester model is the SDOF resonator considered in the examples for this paper. Rather, it holds generally for all harvester models adhering to (12).

However, for the SDOF model, we again have some convenient simplifications with this model, as we did with the unconstrained case in Sec. 3. It turns out that in this case P has the same special structure as in the unconstrained case, i.e., (24), but with P_{22}, P_{24}, and P_{44} defined differently, as

$$P_{22} = \frac{\beta + Y_c^{*2} R}{2\beta + 2Y_c^*} - \frac{1}{2} \tag{52}$$

$$P_{24} = \frac{-Y_c^* + Y_c^{*2} R}{2\left(\beta + Y_c^*\right)\left(\beta + Y_c^* + 2\zeta_a\right)} \tag{53}$$

$$P_{44} = 2\zeta_a \frac{-Y_c^* + Y_c^{*2} R}{2\left(\beta + Y_c^*\right)\left(\beta + Y_c^* + 2\zeta_a\right)} \tag{54}$$

Consequently, our feedback relationship in (51) reduces to

$$i(t) = \underset{vi + i^2/Y_c^{max} \leqslant 0}{\operatorname{sat}} \left\{ -\frac{1}{R}\left(\frac{\beta + Y_c^{*2} R}{2\beta + 2Y_c^*}\right) v(t) - \frac{1}{R}\left(\frac{-Y_c^* + Y_c^{*2} R}{2\left(\beta + Y_c^*\right)\left(\beta + Y_c^* + 2\zeta_a\right)}\right) a(t) \right\} \tag{55}$$

In the equation above, we have again taken advantage of the fact that v and a are the second and fourth components of x.

Recognizing that $i(t) = -Y_c(t)v(t)$, this implies an equation to determine the time-varying $Y_c(t)$ for this controller directly, as

$$Y_c(t) = \underset{[0, Y_c^{max}]}{\operatorname{sat}} \left\{ \frac{1}{R}\left(\frac{\beta + Y_c^{*2} R}{2\beta + 2Y_c^*}\right) + \frac{1}{R}\left(\frac{-Y_c^* + Y_c^{*2} R}{2\left(\beta + Y_c^*\right)\left(\beta + Y_c^* + 2\zeta_a\right)}\right) \frac{a(t)}{v(t)} \right\} \tag{56}$$

The argument in the brackets consists of a constant term, plus a term that varies with the ratio $a(t)/v(t)$. It is this variable term that is responsible for the improvement in performance over the constant Y_c^* case, for the SDOF resonator.

Fig. 4 shows ratios for $\mathcal{E}P_{gen}$ as a function of ζ_a, for the optimal constant Y_c design and for the sub-optimal constrained controller above, each normalized by the performance of the optimal unconstrained controller. The values of $\mathcal{E}P_{gen}$ for the constant Y_c and unconstrained cases can be solved analytically. However, the value of $\mathcal{E}P_{gen}$ for the controller in (56) does not have a closed-form solution, and must be found via simulation. These plots are for a particular set of system parameters (i.e., $\beta = 0.01$, $R = 0.25$, and $Y_c^{max} = 1$) and other parameter combinations give plots with qualitatively similar features.

The main point here is that irrespective of the bandwidth of $a(t)$, the above approach to the adaptation of $Y_c(t)$ results in improvements in $\mathcal{E}P_{gen}$, beyond the optimal power generation with constant $Y_c(t) = Y_c^*$. Typical percentages of improvement beyond the performance with Y_c^* are around 20–30%.

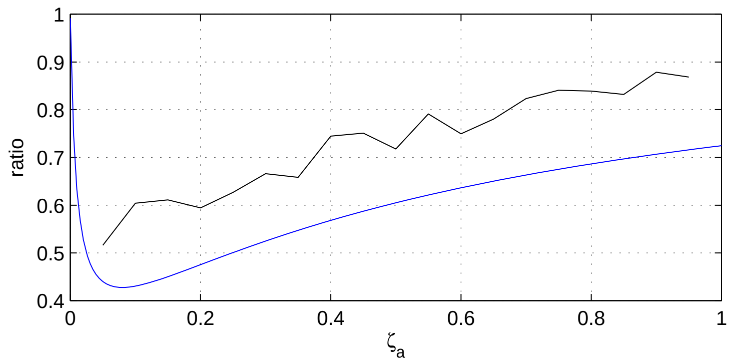


Figure 4. Ratio of $\mathcal{E}P_{gen}$ for optimal constant Y_c (blue) over that of the optimal unconstrained linear current control, and a similar ratio for the nonlinear controller in (56), also normalized by the active performance (black).

5. CONCLUSIONS

The primary purpose of this paper has been to investigate the potential for enhanced energy harvesting performance from stochastic disturbances, through the use of nonlinear admittance adaptation. These results may be of interest in energy harvesting applications for which power flow in the electronics is constrained to flow in only one direction. The main results are contained in Sec. 4.2, which illustrates how the input admittance $Y_c(t)$ to the harvesting electronics can be adapted in response to the concurrent harvester state $x(t)$, to out-perform the best power generation with time-invariant admittance. Furthermore, for the particular SDOF problem considered here, we have derived that this adaptation law only requires knowledge of the harvester voltage $v(t)$ and base acceleration $a(t)$.

ACKNOWLEDGMENTS

This work was supported by NSF award CMMI-0747563. The views expressed in this article are those of the author, and do not necessarily reflect those of the National Science Foundation. The authors also thank Dr. S. Behrens for some helpful and interesting discussions.

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