Modeling and Control of an Electromagnetic Transducer for Vibratory Energy Harvesting Applications

by

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Thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in the Department of Civil and Environmental Engineering in the Graduate School of Duke University

2011
ABSTRACT
(Vibratory Energy Harvesting)

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Abstract

The primary focus of this thesis is on the modeling and control of an electromechanical transducer to harvest energy from large structures (e.g. buildings and bridges). The transducer consists of a back-driven ballscrew coupled to a permanent-magnet synchronous machine. Developing control algorithms to take full advantage of the unique features of this type of transducer requires a mechanical model that can adequately characterize the device’s intrinsic nonlinear behavior. A new model is proposed that can effectively capture this behavior. Comparison with experimental results verifies that the model is accurate over a wide range of operating conditions and that it can be used to correctly design controllers to maximize power generation.

In most vibratory energy harvesting systems the disturbance is most appropriately modeled as a broadband stochastic process. Optimization of the average power generated from such disturbances is a feedback control problem, and the controller can be determined by solving a nonstandard Riccati equation. In this thesis we show that appropriate tuning of passive parameters in the harvesting system results in a decoupled solution to the Riccati equation and a corresponding controller that only requires half of the states for feedback. However, even when the optimal controller requires all of the states for feedback, it is possible to determine the states that contribute the most to the power generation and optimize those partial-state feedback gains using a gradient descent method.

To demonstrate the energy harvesting capability of the transducer, impedance
matching theory is used to optimize power from a small, base-excited single-degree-of-freedom (SDOF) oscillator. For this system, both theoretical and experimental investigations are compared and results are shown to match closely. Finally, statistical linearization is used to determine the optimal full-state controller and the optimal static admittance for the experimental SDOF oscillator when it is excited by a stochastic disturbance.
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List of Symbols and Abbreviations

Symbols

Bolded symbols denote vectors or matrices. Below is a non-exhaustive list of symbols used in this thesis.

A Augmented dynamics matrix.
$A_0$ Acceleration amplitude.
$a$ Disturbance acceleration.
B Augmented control input matrix.
$B$ Rotational viscous damping coefficient.
C Augmented output matrix.
$e$ Electromechanical damping coefficient.
e$_{an}$, $e_{bn}$, $e_{cn}$ Back emf voltages for three stator phases.
f Total transducer force.
f$_c$ Coulomb friction force amplitude.
f$_e$ Electromechanical force.
G Augmented exogenous disturbance input matrix.
$\mathcal{H}$ Hamiltonian matrix.
i Control current.
i$_a$, i$_b$, i$_c$ Electrical currents in three stator phase windings.
J Rotational inertia coefficient.
\( K \)  Optimal full-state gain matrix.
\( \tilde{K} \)  Optimal partial-state gain matrix.
\( K_e \)  Back emf voltage magnitude.
\( L \)  Luenberger observer gain matrix.
\( L_c \)  Stator coil inductance.
\( l \)  Ballscrew lead.
\( \tilde{P}_{gen} \)  Average power generation.
\( \tilde{P}_L \)  Average power generated across a resistive load.
\( R \)  Effective resistance of the electronics.
\( R_c \)  Stator coil resistance.
\( R_L \)  Load resistance.
\( r \)  Harvester relative displacement.
\( v \)  Harvester voltage.
\( v_{an}, v_{bm}, v_{cn} \)  Line-to-neutral voltages for stator coils.
\( w \)  White noise process.
\( x \)  State vector.
\( Y_c \)  Dynamic admittance.
\( Y_0 \)  Static admittance.
\( \alpha \)  Ratio of the TMD mass to the structure mass.
\( \delta \)  Dirac delta function.
\( \zeta_a \)  Disturbance filter damping ratio.
\( \zeta_0 \)  Viscous damping ratio.
\( \theta_m \)  Angular position of the rotor.
\( \Sigma \)  Covariance matrix.
\( \sigma_a \)  Disturbance acceleration standard deviation.
\( \Phi_a \)  Disturbance acceleration power spectral density.
\( \omega \)  Disturbance filter natural frequency.
\( \nabla \)  Gradient operator.
\( \mathcal{E} \)  Expectation operator.

Abbreviations

Below is a non-exhaustive list of abbreviations used in this thesis.

- **emf**: Electromotive force.
- **LQG**: Linear-quadratic-Gaussian.
- **pdf**: Probability distribution function.
- **RTHT**: Real-time hybrid testing.
- **SDOF**: Single-degree-of-freedom.
- **TMD**: Tuned mass damper.
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1 Introduction

1.1 Background and Motivation

Electromechanical systems to harvest vibratory energy have been the subject of considerable research activity over the last decade. An example of a typical vibratory energy harvesting system can be seen in Figure 1.1. As shown, energy conversion is accomplished by a passive electromechanical system, which consists of flexible

![Diagram of a typical vibratory energy harvesting system.](image)

**Figure 1.1**: A typical vibratory energy harvesting system.
mechanical structure with an embedded transducer to generate power from an exogenous disturbance. The transducer must be interfaced with a power electronic network that extracts power and delivers it to a storage device, such as a rechargeable battery or a supercapacitor. Constraints on the directionality of power flow depends on the type of power electronic network that is attached to the transducer. Generally, the power electronic network is controlled by a microprocessor or analog circuit whose feedback algorithm makes power generation decisions based on measurements of various system properties (i.e., displacement, acceleration, transducer voltage, etc.).

The focus of energy harvesting research has been primarily on low-power applications requiring energy-autonomy, such as wireless sensing and embedded computing systems (Roundy et al., 2002; Beeby et al., 2006; Anton and Sodano, 2007; Knight et al., 2008). Most of this research has been motivated by vibration applications in the aerospace sector and is focused on vibratory regimes characterized by frequencies above 10Hz, displacement oscillations on the order of millimeters, and power on the scale of milliwatts (or below). For such applications, piezoelectric transducers have been the dominant technology, although electrostatic and electromagnetic MEMS transducers (Mitcheson et al., 2004; Amirtharajah and Chandrakasan, 1998) have also seen significant attention.

In contrast to these applications, the focus of this thesis will be on the modeling and control of an electromagnetic transducer to harvest energy from mechanical vibrations from large structures (e.g., buildings and bridges). The vibratory regime of these structures oscillating in their fundamental mode is typically around 1Hz and the available power can be on the kilowatt scale. Transducers designed to extract this energy must therefore be capable of extracting significant power from low-frequency, low-velocity oscillations. Operation in this vibratory regime requires very different electromechanical hardware than in higher-frequency, lower-power applications.
Large-scale electromagnetic transduction in vibrating structures has a number of useful applications. Most immediate among these is their use in large-scale energy harvesting applications, in which the vibratory energy in a structure, when excited by its surrounding environment, is converted and stored for use by electronic subsystems. Such environmental disturbances may include, for example, wind excitations on buildings (Ni et al., 2011), wave excitations on offshore structures (Scruggs and Jacob, 2009), traffic-induced vibrations in bridges, and seismic events (Scruggs and Iwan, 2003, 2004). Such systems also have application in vehicles, including automotive suspensions (Zuo et al., 2010) and railway systems (Nagode et al., 2010). The energy harvested from these vibrations can be used to power wireless sensing networks and other intelligent systems for structural health monitoring (Glynne-Jones et al., 2004). It can also be used to provide parasitic power for semiactive structural control devices, such as magnetorheological dampers (Choi and Wereley, 2009).

1.2 Electromagnetic Transducer Description

The primary focus of this thesis is on the modeling and control of an electromagnetic transducer capable of generating over 100W of power from sub-Hertz structural vibration with a force rating of 1kN. While this system is a scaled-down version of a system which would be appropriate for full-scale civil structure applications, it serves as a bench-scale demonstration of the technology. Figure 1.2 provides an illustration of the electromagnetic transducer that will be studied in this thesis. The basic conversion system consists of a back-driven precision ballscrew, which is coupled to the shaft of a three-phase, permanent-magnet synchronous machine. Such devices (as well as planetary roller screws) constitute one of the most efficient methods of linear-to-rotational conversion when power flow is in the direction from linear to rotational motion, and when high mechanical advantage is important. The machine terminals
Evaluating the usefulness of a ballscrew transducer to harvest energy from large-scale structures requires the development of a high-fidelity model for use in feedback design and analysis. This task is challenging because the device is nonlinear. Several studies have developed analogous approaches to the one taken in this thesis, for characterizing the nonlinearities in magnetorheological (MR) devices (Spencer et al., 1997; Yang et al., 2002) as well as in electrorheological (ER) devices (Gamota and Filisko, 1991; Kamath et al., 1996; Gavin et al., 1996) for semi-active control applications. The nonlinearities that arise in MR and ER dampers are mainly due to the yielding stress of the fluid. However, the nonlinearities present in the proposed device are a result of the sliding friction of the ballscrew. In addition, hysteresis in the force-velocity plane is observed, which can be attributed to the elasticity of the timing belt that connects the ballscrew to the shaft of the synchronous machine. Considering these effects, a new predictive model is proposed that is numerically tractable and effectively portrays the behavior of the ballscrew device.
Design techniques for linear electromagnetic energy harvesters have been studied by Zuo et al. (2010) for vehicle suspension applications. In that study, it is shown that the magnetic flux intensity of the device can be improved using finite element modeling of the coil and magnet assemblies. Additional studies (Nerves and Krishnan, 1996; Palomera-Arias et al., 2008) have examined the capability of electromagnetic transducers to be used as dampers by dissipating the absorbed energy in various electrical subsystems. In particular, the study by Palomera-Arias et al. (2008) provides a detailed theoretical model for a linear electromagnetic transducer, which could be used to design similar transducers for structural control or energy harvesting applications.

Another potential application for electromagnetic transducers is to harvest energy from tuned mass dampers (TMDs) in tall buildings that are excited by wind (Tang and Zuo, 2011; Ni et al., 2011). It is estimated in Ni et al. (2011) that the amount of power that is dissipated by a TMD in a tall building is on the order of kW–MW. One of the simplest ways to optimize the absorbed power from such an application is by determining the impedance of the harvesting transducer that maximizes power generation. Impedance matching techniques for electromechanical systems are well understood in the literature (Stephen, 2006). Several studies (Lefeuvre et al., 2007; Kong et al., 2010) have utilized power electronic drives to simulate impedance matched loads for piezoelectric energy harvesting applications. In addition, the study by Ward and Behrens (2008) investigates adaptively updating the optimal admittance of an electromagnetic energy harvester that was subjected to step disturbances. The adaptive admittance was determined through machine learning and was experimentally implemented via a four-quadrant bridge rectifier.
1.3 Power Electronic Converters for Energy Harvesting

The efficient extraction and transmission of the electrical power from an energy harvesting system to a storage device (i.e., a rechargeable battery or a supercapacitor) has become the subject of considerable research over the past decade. At the very least, the AC power that is harvested from an energy harvesting transducer must be rectified through a standard diode bridge before it can be delivered to storage. However, the theoretical rate of power extraction can be greatly improved by interfacing the harvesting transducer with a power electronic switching converter. Power electronic converters are capable of imposing static or dynamic relationships between the voltages and currents of the transducer by operating the transistors like switches.

One of the most popular power electronic converters for energy harvesting applications is the single-directional DC/DC converter in Figure 1.3(a). This converter, and variants thereof, have become the subject of considerable interest for low-power energy harvesting applications (Kasyap et al., 2002; Lefeuvre et al., 2007; Kong et al., 2010). This particular circuit is called a buck-boost converter due to the fact that it can make the output voltage magnitude less than or greater than the input voltage magnitude. Power flow in such converters is regulated via high-frequency

![Figure 1.3](image_url)

**Figure 1.3:** Energy harvesting converters interfaced with a storage bus. (a) A full bridge rectifier connected to a buck-boost converter, (b) an h-bridge.
pulse-width modulation (PWM) switching control of the MOSFETs in the circuitry. Single-directional DC/DC converters are advantageous in small-scale applications because they only require that a single MOSFET be switched, thus reducing gating losses. However, for applications where the parasitic losses are small in comparison to the average power generated, the single-directional power flow capability of these converters can hamper their ability to optimize power generation from stochastic dynamical responses, as illustrated by Scruggs (2010).

For energy harvesting applications in which the disturbance can be characterized by a sinusoid at a single frequency, such converters can be incorporated into a larger passive network which presents an optimal admittance at the transducer terminals. As pointed out in (Stephen, 2006; Scruggs, 2010), power generation is optimized by matching the input admittance of the effective passive network (including the effective resistance of the converter) to the complex conjugate transpose of the harvester admittance at this frequency. Due to this result, several researchers have investigated impedance matching techniques by operating various DC/DC converters in the discontinuous conduction regime (Ottman et al., 2003; Lefeuvre et al., 2007; Kong et al., 2010). In this regime, the inductor fully demagnetizes (i.e., its current drops to zero) before the end of the switching cycle, and remains so until the converter’s MOSFET is gated on again at the leading edge of the next switching cycle. Operation of a buck-boost converter in discontinuous conduction results in an input admittance which is decoupled from the behavior of the storage voltage. Furthermore, if the transducer side capacitance, $C_t$, is sufficiently small, and possibly with supplemental input filtering, then the admittance is approximately linear and resistive in low- to mid-frequency bands. As such, the design of these converters for optimal operation proceeds by first determining the effective static admittance which maximizes power generation from the transducer, and then synthesizing the duty cycle from this resistance.
However, for cases in which the disturbance is modeled as a stochastic process, optimization of the dynamic behavior of the electronics for maximal power generation is more challenging. It has been shown in Scruggs (2010) that maximizing the power generated from an energy harvester using the same impedance matching techniques mentioned above, always results in an anti-causal admittance. That study also showed that if the electronics of energy conversion are efficient enough, then the optimal causal control of transducer current (as derived by LQG theory) cannot be realized with a passive network. This is because in such circumstances, there are frequency bands in which the average energy for the optimized system flows in the reverse direction; i.e., from storage back into the harvester.

The realization of a synthetic dynamic admittance to maximize power generation from a stochastic disturbance can be accomplished by a two-way (i.e., four-quadrant) power electronic converter. One example of such a converter, called an H-bridge, is shown in Figure 1.3(b). Operation of the H-bridge requires control of four MOSFETs and thus requires more parasitic power than a one-directional converter, which is the price paid for two-way directionality of power flow. Many previous studies have used H-bridge drives for various piezoelectric applications (Main et al., 1996; Chandrasekaran et al., 2000), including their proposed use in piezoelectric energy harvesting applications (Liu et al., 2009; Scruggs, 2009). H-bridge control of currents for electromagnetic transducers is standard, and their use in electromagnetic energy harvesting applications has been investigated by Ward and Behrens (2008). Irrespective of the application, the drive has to be capable of high-bandwidth current tracking. This technology is well understood and can be accomplished using hysteretic switching or PWM techniques (Kassakian et al., 1991).

Because the transducer used in this thesis relies on a three-phase, permanent-magnet synchronous machine for power conversion, the power electronic converters in Figure 1.3 would have to be augmented to account for the additional phase. The
converter in Figure 1.3(a) would include an additional half-bridge diode rectifier while the converter in Figure 1.3(b) would include an addition half H-bridge. Extending the theory used to determine the optimal static or dynamic admittance that would be imposed by the three-phase converter is also straightforward. It is possible to express the dynamics of the three-phase transducer as an equivalent single phase device using Park coordinates (Krishnan, 2001).

1.4 Objectives of this Thesis

The first objective of this thesis is to provide an approach to the system-level design of an electromagnetic transducer. The design process will account for the interdependency of the electronics, electronic controller, machine parameters, mechanical conversion system, and the characteristics of the structure into which the system is to be embedded. Four dimensionless design metrics are developed and are evaluated for the transducer used in the experimental portion this thesis. Recommendations in terms of minimal threshold values are made for each design metric.

The second objective of this thesis is to derive a controller to maximize the average power generated by an electromagnetic energy harvester that is subjected to a broadband stochastic disturbance. Building upon the techniques developed by Scruggs (2009, 2010) and the results presented in Scruggs and Cassidy (2010), this thesis shows that for stochastic disturbances characterized by second-order, bandpass-filtered white noise, energy harvesters can be passively “tuned” such that optimal stationary power generation only requires half of the system states for feedback in the active circuit. Interestingly, these states are the ones most easy to sense. For example, for a base-excited SDOF electromagnetic harvester, only the disturbance acceleration and transducer voltage are needed for feedback. One can view these tuning techniques as a “stochastic counterpart” to the tuning techniques used in harvesting applications characterized by deterministic disturbances. In such cases,
electrical tuning techniques can be used to maximize harvested power at the excitation frequency using only transducer voltage feedback, resulting in a harvesting circuit with a resistive input impedance. By contrast, optimal power generation in a stochastic context, even with the use of tuning techniques, will almost always require knowledge of other quantities in addition to transducer voltage.

The fact that only half the states are required for feedback in tuned harvesters is a by-product of a special structure which arises in the matrix solution to the associated Riccati equation used to synthesize the optimal feedback gains. In effect, the passive tuning techniques have the effect of causing many of the entries in this matrix to be identically zero. However, in many applications such tuning techniques may be impractical, due, for example, to the large inductances they often require for piezoelectric applications. In other applications, even with the tuning techniques implemented as described, it may not be practical to feed back the reduced subset of system states required for optimality. Motivated by this observation, the third objective of this thesis is to use fixed-structure feedback optimization techniques to maximize power generation for the case in which the availability of various system states for feedback is fixed in the design. For this case, we assume that the transducer current is determined based only on concurrent measurements of a set of prescribed states (i.e., the feedback law is presumed to be static), resulting in a decision process which can be realized with a very simple and efficient analog circuit. This makes the results of relevance even in small-scale electromechanical transducers where microprocessors and elaborate control algorithms are no longer viable.

The fourth objective of this thesis focuses on the derivation of an analytical expression for the average power generated by the transducer when the terminals of the motor are connected to resistive shunts. This expression can be used to determine the impedance matched resistive load for the transducer which is embedded within a single-degree-of-freedom (SDOF) oscillator, which is meant to represent a tuned mass
damper (TMD) in a much larger structure. Our approach to designing the matched
impedance takes the identified viscous damping and Coulomb friction present in the
transducer into account. The analytical expression for average power generated is
compared to our experimental system in which a SDOF oscillator is simulated using
hybrid testing.

Finally, the fifth objective of this thesis is on derivation of the optimal full-state
feedback controller for the experimental energy harvesting system that is subjected
to a stochastic disturbance. To accomplish this objective, we exploit the stationarity
properties of a stochastic system in order to determine the statistically linearized
covariance matrix, which accounts for the Coulomb friction present in the transducer.
Statistical linearization is a technique which approximates the effects of nonlinearities
in a dynamic system that is excited by white noise (Roberts and Spanos, 2003).
Using this technique and the necessary conditions from optimal control theory, we
derive the optimal full-state feedback controller and the optimal static admittance
for a SDOF oscillator with Coulomb friction.
In this chapter, detailed models of the electrical and mechanical dynamics of the experimental device are developed. Beginning with the first principles that describe the electrical dynamics of a three-phase permanent-magnet synchronous machine, we derive equations that describe the electrical dynamics of the device. The electrical dynamics are simplified by expressing the voltages and currents in terms of Park coordinates (Pillay and Krishnan, 1989). From these simplified expressions, we derive an expression for the electromagnetic force that can be applied by the device when the terminals of the motor are connected to resistive shunts. Next, we present four design tradeoffs between the various subsystems (including the controller, electronics, machine, mechanical conversion, and structural system) and an approach to device optimization is presented.

In addition to the electrical model, we present a physically-motivated mechanical model to describe the nonlinear dissipative phenomena present in the device. This model accounts for oscillations in the force-displacement plane, which is caused by sliding friction effects of the ballscrew, and hysteresis in the force-velocity plane, which is a result of the elastic belt that connects the ballscrew to the shaft of the
motor. Parameters in the model are optimized using a Levenberg-Marquardt least squares algorithm. Once these parameters have been optimized for a sine-sweep test over a range of frequencies, we compare the predicted response of the device to experimental measurements for various operating conditions.

2.1 Device Modeling

A picture of the experimental setup consisting of the electromagnetic transducer used in this chapter is shown in Figure 2.1. Linear-to-rotational conversion is accomplished via a precision ballscrew. The particular ball screw mechanism used here is a Thomson EC3 unit (Kollmorgen, 2010b). It has a lead of 16mm/rev, and a diameter of 16mm, resulting in a helical angle of approximately 17 degrees. The force rating on the screw is 7.2kN, well in excess of the force range for the experiments conducted in this chapter. The ballscrew is interfaced with the motor shaft via a timing belt with a 1:1 ratio. The configuration shown, in which the motor is placed in tandem with the screw, was chosen because it allowed for easier mounting of the device at either end via clevis brackets. However, a custom configuration in which the motor shaft is mounted directly to the rotating screw, and which also permits a clevis bracket to be mounted behind the motor housing, would reduce the need for the timing belt and also reduce Coulomb friction in the system by reducing side-loads on both rotating
shafts.

The linear velocity $\dot{x}$ of the device is related to its mechanical angular velocity $\dot{\theta}_m$ via the lead conversion; i.e.,

$$\dot{x} = l \dot{\theta}_m$$

(2.1)

where $l$ is in m/rad. The linear-to-rotational conversion of the ballscrew can be modeled as relating the linear force $f$ to the electromechanical force $f_e$ of the motor, via the equation

$$f = f_e + f_b - \frac{J}{l^2} \ddot{x} - \frac{B}{l^2} \dot{x}$$

(2.2)

where the sign convention is that $f$ and $\dot{x}$ have the same directional sense. In the above expression, $B$ and $J$ are the rotational viscosity and inertia, respectively, of the combined motor shaft and screw, and $f_b$ is the nonlinear damping force associated with the bearing friction of the ballscrew. For the remainder of this study we will express the total device force as

$$f = f_e + f_d$$

(2.3)

where $f_d$ includes the inertia, viscous damping, and bearing friction of the ballscrew as well as other characteristics of the device. A detailed model describing this term is further discussed in Section 2.3.

The motor itself is a Kollmorgen AKM42E (Kollmorgen, 2010a) three-phase permanent-magnet synchronous motor with polarity $p = 5$ (Krishnan, 2001). An illustration of the cross section of a typical three-phase permanent-magnet synchronous machine can be seen in Figure 2.2(a). The choice of a rotary three-phase AC motor (as opposed to a rotary DC motor or a linear motor) was made based on practical considerations. Generally, for motors with comparable power ratings, ones with higher velocity (and lower torque) capability tend to be less massive, as they require less iron to sustain the necessary magnetic field. This becomes even more pronounced
Figure 2.2: (a) Cross section of a typical three-phase permanent-magnet synchronous machine; (b) equivalent electrical schematic.

in the comparison between rotary and linear motors of comparable power ratings. Additionally, rotary synchronous three-phase machines are almost always more efficient as power generators, because they can generate a larger back-emf for a given mechanical velocity, while also possessing a lower coil resistance.

The internal voltages (i.e., back-emfs) for phases $a$, $b$, and $c$, relative to the neutral node $n$, can be expressed as a function of the mechanical angle $\theta_m$, as

$$
e_{an} = K_e \dot{\theta}_m \cos (p \theta_m) \quad (2.4a)$$

$$
e_{bn} = K_e \dot{\theta}_m \cos \left( p \theta_m + \frac{2\pi}{3} \right) \quad (2.4b)$$

$$
e_{cn} = K_e \dot{\theta}_m \cos \left( p \theta_m - \frac{2\pi}{3} \right) \quad (2.4c)$$

where $K_e$ is the back-emf constant. (See Figure 2.2(b) for a schematic.) Let the line-to-neutral voltages at the terminals of the three phases be $v_{an}$, $v_{bn}$, and $v_{cn}$. Ultimately, these voltages will be imposed by the circuitry to be attached to the motor. Then the line current into phase $a$ evolves according to

$$
\frac{d}{dt}i_a = \frac{1}{L_c} \left( -R_c i_a + v_{an} - e_{an} \right) \quad (2.5)
$$

where $L_c$ and $R_c$ are the line-to-neutral coil inductance and coil resistance, respec-
tively, of the motor. Analogous equations hold for the $b$ and $c$ phases. For the remainder of this chapter $L_c$ and $R_c$ are assumed to be the same for each of the phases (this was confirmed experimentally). Thus, in matrix form, we have that the equation describing the electrical dynamics of the motor is

$$\frac{d}{dt}i_{abc} = \frac{1}{L_c} \left( -R_c i_{abc} + v_{abc} - e_{abc} \right),$$

(2.6)

where $i_{abc}$, $v_{abc}$, and $e_{abc}$ are vectors of the line-to-neutral currents, voltages, and back-emfs, respectively.

It is convenient to express the dynamics of a three-phase machine in terms of quadrature ($q$), direct ($d$), zero ($0$) coordinates, via the Park transformation (Pillay and Krishnan, 1989). We start by defining the vector of $q$, $d$, and 0 currents as a linear transformation from the line currents; i.e.,

$$i_{qd0} = P(\theta_m) i_{abc}$$

(2.7)

where

$$P(\theta_m) = \begin{bmatrix} \cos(p\theta_m) & \cos(p\theta_m + \frac{2\pi}{3}) & \cos(p\theta_m - \frac{2\pi}{3}) \\ \sin(p\theta_m) & \sin(p\theta_m + \frac{2\pi}{3}) & \sin(p\theta_m - \frac{2\pi}{3}) \end{bmatrix}.$$ 

(2.8)

Similarly, for the $q$, $d$, and 0 voltages,

$$v_{qd0} = P(\theta_m) v_{abc}.$$ 

(2.9)

Substituting Equations (2.7) and (2.9) into Equation (2.6) gives

$$\frac{d}{dt}i_{qd0} - p\dot{\theta}_m Q_i_{qd0} = \frac{1}{L_c} \left( -R_c i_{qd0} + v_{qd0} + P(\theta_m)e_{abc} \right)$$

(2.10)
where the matrix \( Q \) can be found as
\[
Q = \left( \frac{\partial}{\partial \theta_m} P(\theta_m) \right) P^{-1}(\theta_m) = \begin{bmatrix}
-p \sin(p\theta_m) & -p \sin \left( p\theta_m + \frac{2\pi}{3} \right) & -p \sin \left( p\theta_m - \frac{2\pi}{3} \right) \\
p \cos(p\theta_m) & p \cos \left( p\theta_m + \frac{2\pi}{3} \right) & p \cos \left( p\theta_m - \frac{2\pi}{3} \right) \\
0 & 0 & 0
\end{bmatrix}
\]
\[
\times \frac{2}{3p} \begin{bmatrix}
\cos(p\theta_m) & \sin(p\theta_m) & 1 \\
\cos(p\theta_m + \frac{2\pi}{3}) & \sin \left( p\theta_m + \frac{2\pi}{3} \right) & 1 \\
\cos(p\theta_m - \frac{2\pi}{3}) & \sin \left( p\theta_m - \frac{2\pi}{3} \right) & 1
\end{bmatrix}
\]
(2.11)
\[
= \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
(2.12)

We now have three nonlinear differential equations that describe the dynamics of the motor; i.e.,
\[
\frac{d}{dt} i_q = \frac{1}{L_c} \left( -R_c i_q + v_q - p \dot{\theta}_m L_c i_d + \frac{3K_e}{2} \dot{\theta}_m \right)
\]
(2.13)
\[
\frac{d}{dt} i_d = \frac{1}{L_c} \left( -R_c i_d + v_d + p \dot{\theta}_m L_c i_q \right)
\]
(2.14)
\[
\frac{d}{dt} i_0 = \frac{1}{L_c} \left( -R_c i_0 + v_0 \right).
\]
(2.15)

The neutral node \( n \) is inaccessible, and because of this, external voltages can only be applied from line-to-line. This results in the constraint \( v_0 = \frac{1}{2} \left( v_{an} + v_{bn} + v_{cn} \right) = 0 \). In the equations above, this results in \( i_0 = 0 \), which may be alternatively confirmed by Kirchhoff’s current summation law at the neutral node. Thus, Equation (2.15) can be eliminated from the dynamical description of the system, which is fully characterized by Equations (2.13) and (2.14). Finally, we have that the electromechanical force that can be applied by the device is proportional to the quadrature current and can be expressed as
\[
f_e = -\frac{3K_e}{2l} i_q.
\]
(2.16)

We now consider the case in which the transducer will ultimately be used in an application as shown in Figure 2.3. As shown, the transducer is attached to a single-
degree-of-freedom (SDOF) oscillator with mass $m$, damping $c$, and stiffness $k$. One of the simplest ways to harvest power from this application is to have the electronics simulate resistive loads $R_L$ across the terminals of the motor. This can be realized by simply attaching three resistive loads in a star connection to the motor, which is illustrated in Figure 2.3. With this configuration, we have that $\mathbf{v}_{abc} = -R_L \mathbf{i}_{abc}$, and consequently $\mathbf{v}_{q0} = -R_L \mathbf{i}_{q0}$. As such, the equations describing the $i_q$ and $i_d$ can be expressed as

\begin{align}
\frac{d}{dt} i_q &= \frac{1}{L_c} \left( - (R_c + R_L) i_q - p \dot{\theta}_m L_c i_d + \frac{3K_e}{2} \dot{\theta}_m \right) \quad (2.17) \\
\frac{d}{dt} i_d &= \frac{1}{L_c} \left( - (R_c + R_L) i_d + p \dot{\theta}_m L_c i_q \right). \quad (2.18)
\end{align}

We next make the approximation that $i_d \approx 0$ under normal operating conditions. This assumption can be justified as follows. If we assume that $\dot{\theta}_m$ is slowly-varying, then we can solve for the $\dot{\theta}_m$-dependent equilibrium of $i_q$ and $i_d$. We start by assuming that the derivatives of $i_q$ and $i_d$ are equal to zero. As such, we have that Equations (2.17) and (2.18) can be written in matrix form as

\begin{equation}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -(R_c + R_L)/L_c & -p \dot{\theta}_m \\ p \dot{\theta}_m & -(R_c + R_L)/L_c \end{bmatrix} \begin{bmatrix} i_q \\ i_d \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{3K_e}{2L_c} \end{bmatrix} \dot{\theta}_m. \quad (2.19)
\end{equation}

Thus, we can solve for the $\dot{\theta}_m$-dependent equilibrium of $i_q$ and $i_d$ through the following
calculation

\[
\begin{bmatrix}
    i_q \\
    i_d
\end{bmatrix} = \frac{1}{\left(\frac{R_c + R_L}{L_c}\right)^2 + \left(p\dot{\theta}_m\right)^2} \begin{bmatrix}
    -(R_c + R_L)/L_c \\
    -p\dot{\theta}_m \\
    -\frac{3K_e}{2L_c} \dot{\theta}_m
\end{bmatrix}
\] 

\begin{equation}
\begin{bmatrix}
    i_q \\
    i_d
\end{bmatrix} = \frac{3K_e\dot{\theta}_m}{2 \left(\frac{R_c + R_L}{L_c}\right)^2 + 2 \left(L_c p\dot{\theta}_m\right)} \begin{bmatrix}
    (R_c + R_L) \\
    L_c p\dot{\theta}_m
\end{bmatrix}.
\end{equation}

From the expression in Equation (2.21), we have that the dynamics of \(i_d\) can be neglected as long as \(p\dot{\theta}_m \ll (R_c + R_L)/L_c\). For the experiments presented in this chapter, the load resistance \(R_L\) will never be below the coil resistance \(R_c\). As such, if we conservatively make \(R_L = R_c\), then, in terms of the linear velocity of the device, the dynamics of \(i_d\) can be neglected as long as \(\dot{x}\) is well below 27.2cm/s. If the device were run at higher velocities, then vector control (Pillay and Krishnan, 1989) using velocity feedback should be used to explicitly control \(i_d\) to be zero. Although this would complicate the conceptual approach illustrated in Figure 2.3 (because the harvested currents would be controlled in a manner which is more complicated than a simple star-connected resistive impedance), the derived model would not change.

Using the simplifications that \(i_0 = 0\) and \(i_d = 0\), we have that the differential equation describing \(i_q\) is

\[
\frac{d}{dt}i_q = \frac{1}{L_c} \left( -(R_c + R_L)i_q + \frac{3K_e}{2} \dot{\theta}_m \right) .
\] 

\begin{equation}
\frac{d}{dt}i_q = \frac{1}{L_c} \left( -(R_c + R_L)i_q + \frac{3K_e}{2} \dot{\theta}_m \right) .
\end{equation}

This expression is analogous to the electrical dynamics of a single phase DC machine.

Next, we substitute

\[
i_q = -\frac{v_q}{R_c + R_L}
\]

\begin{equation}
i_q = -\frac{v_q}{R_c + R_L}
\end{equation}

into Equation (2.16), and we have that at low frequencies, where \(i_q\) may be assumed to be a slowly-varying function of \(\dot{\theta}_m\), the electromechanical force can be expressed.
as a function to the linear velocity of the device; i.e.,

$$f_e = \frac{9K_e^2}{4l^2(R_c + R_L)} \dot{x} = c_e \dot{x}. \quad (2.24)$$

We can think of $c_e$ as the equivalent electromechanical viscous damping term associated with connecting resistive loads across the terminals of the motor. From the expression in Equation (2.24), we can express the total electromechanical power that is absorbed by the system as

$$P_a = f_e \dot{x} = \frac{9K_e^2}{4l^2(R_c + R_L)} \dot{x}^2. \quad (2.25)$$

Harvesting power from the resistive load configuration in Figure 2.3 can be accomplished by connecting the three-phase motor to a servo drive, which interfaces the machine with a DC power bus (also called a “DC link”). The block diagram in Figure 2.4 demonstrates this concept. The particular three-phase servo drive that is used in this chapter is an S16A8 analog servo drive from Advanced Motion Controls (Advanced Motion Controls, 2010). The servo drive requires command signals for

![Figure 2.4: Block diagram of the three-phase motor interface with an analog servo drive.](image-url)
two of the phases (denoted by $i_a^*$ and $i_b^*$) and determines the command for the third phase by enforcing the fact that the sum of the line currents is equal to zero. Tracking of the desired command signals is accomplished through high-frequency pulsewidth modulation (PWM) switching control of six MOSFETs. The switching frequency of the MOSFETs has been set at 33kHz. Line-to-neutral voltages of two of the phases are sensed with analog differential amplifiers and the output voltage measurements are then sent to a dSpace DS1103 data acquisition system. In dSpace, the line-to-neutral voltages are divided by the desired load resistance and the resulting current command signals are sent to the servo drive.

One of the main advantages of using a drive of this nature is that it is capable of four quadrant regenerative operation. However, there are several drawbacks to using an off-the-shelf drive for structural vibration applications. The first is the fact that the switching frequency of the MOSFETs is 33kHz and there is no way to adjust this setting. This switching frequency is more than an order of magnitude higher than what is necessary to track the desired currents that the drive will impose on the motor. For energy harvesting applications, it has been shown in Ottman et al. (2003) that there is an optimal switching frequency of power electronic drives at which harvested power is maximized. Another drawback is due to the way in which the supply power is converted into logic power. An analog step-down converter is used to reduce the supply voltage $V_S$ (which can range from 60–80V) to 12V to power various drive intelligence components. During this process a significant amount of power is being dissipated. Both of these drawbacks result in significant parasitic power losses, which can be reduced by using a drive that customized for the specific application.

To demonstrate the force capability of the device, the servo drive is used to simulate various resistive loads for a sinusoidal displacement input. The device is back-driven with 0.1Hz displacement with an amplitude of 7.5cm and the results...
are presented in Figure 2.5. Resistance values of 30Ω, 9Ω, and 3Ω in addition to the open circuit case were chosen to illustrate the full force range of the device at this velocity. It should be noted that the hysteresis plots in Figure 2.5(b) appear to exhibit predominately viscous damping behavior, with a Coulomb friction force threshold. The high frequency oscillations that are present in each test are due to the bearing friction in the ballscrew. A detailed model describing these effects is presented in Section 2.3. Resistance control could be used to compensate for the oscillations in the hysteresis plots, due to spatially-dependent Coulomb friction. However, this issue is beyond the scope of this chapter, which is limited to a basic characterization and demonstration of the device used in this thesis.

Figure 2.5: Experimentally measured transducer force for various load resistances simulated by the servo drive: (a) force versus time; (b) force versus displacement; (c) force versus velocity.
2.2 Design Tradeoffs

The efficacy of the type of electromagnetic transducer, as described here, will depend quite heavily on the combination of parameters for the screw conversion, motor, electronics, and the characteristics of the SDOF oscillator. It is important that all these components be taken into account in the design, especially because the combinations of parameters leading to the transducer’s effective use as a generator are very different from those that would be chosen to operate it as an actuator for position tracking, which is its conventional usage. In particular, there are several criteria which should be met for the device to operate as intended.

- **Damping capability:** The quantity

\[
\varepsilon_{\text{short}} = \frac{9K_e^2}{4R_c\nu^2} \tag{2.26}
\]

is approximately equal to the effective linear viscous damping contributed by the motor, when its terminals are shorted (i.e., when \(v_{an} = v_{bn} = v_{cn} = 0\)). When multiplied by the square of the linear velocity, this gives the maximum rate at which mechanical power can be extracted by the device (at a given velocity), before the internal dissipation in the motor exceeds power extraction. It is therefore the quantity most closely connected with the efficiency with which a combination of screw conversion (i.e., \(l\)) and machine (i.e., \(K_e\) and \(R\)) can manage a given amount of mechanical power. A good indicator of whether the hardware is properly sized for the application is therefore a measure of the fraction of critical damping imposed on the SDOF oscillator with this maximum viscosity imposed; i.e.,

\[
\zeta_{\text{max}} = \frac{\varepsilon_{\text{short}}}{2\sqrt{km}}. \tag{2.27}
\]

For effective energy harvesting, the effective linear viscous damping imposed
by the device should be considerably below $\zeta_{\text{short}}$. Therefore, the fraction of critical damping desired for the SDOF oscillator should be considerably below $\zeta_{\text{max}}$. As such, we define the first performance index for the device as

$$I_c = \zeta_{\text{max}}$$

(2.28)

with high values being more favorable. If the transducer is being used in structural control applications, $\zeta_{\text{max}}$ should be made as high as possible to give the device a wide range of variable damping.

- **Force capability**: The maximum force rating for the entire device will be approximately

$$f^\text{rat} = \min\{f^\text{rat}_{\text{screw}}, T^\text{rat}_{\text{e}}\}$$

(2.29)

where $f^\text{rat}_{\text{screw}}$ is the rated axial load of the screw and $T^\text{rat}_{\text{e}}$ is the rated torque of the machine. This force should be sizable enough to have significant influence over the dynamics of the oscillator. So, for example, if the SDOF oscillator is excited by a base excitation near resonance with acceleration amplitude $A_0$, then a helpful index for the suitability of the device is the fraction

$$I_f = \frac{f^\text{rat}}{mA_0}.$$ 

(2.30)

Clearly, a higher value of $I_f$ affords more control capability over the mass.

- **Velocity amplification**: Consider the case in which the servo drive in Figure 2.4 is replaced by a full bridge diode rectifier. The diodes have a conduction voltage $v_d$, which must be overcome by the back-emfs, before power can be transferred from the machine to the DC bus. This voltage threshold is almost unavoidable in power-electronics (unless the diodes are replaced with mechanical relays), and will persist even if more sophisticated electronics are used to
control power extraction. As such, for the typical velocities to be experienced by the transducer, the voltages need to exceed this threshold by a comfortable margin. To evaluate this, we need to specify a design velocity, $\dot{x}_0$, for the transducer. Then, we define the index

$$I_v = \frac{3K_v\dot{x}_0}{2v_d l}.$$  

(2.31)

This fraction is equal to 1 if the back-emfs are just large enough to result in continuous conduction through the diode rectifier for their entire oscillatory cycle. Clearly, a suitable device should have $I_v \gg 1$.

- **Inertia suppression:** The last criterion has to do with the fact that there is a finite rotary inertia $J$ in the system, and if the lead $l$ is too small, this inertia will be reflected into the linear dynamics as an equivalent mass $J/l^2$, sizable enough so as to hamper the transducers dynamics. Indeed, this effect can often be the limiting factor in the choice of lead $l$. Thus, the index

$$I_m = \frac{ml^2}{J}.$$  

(2.32)

is critical to the suitability of the transducer, and design parameters should be chosen such that $I_m \gg 1$.

For example, the transducer in Figure 2.1 has relevant properties as listed in Table 2.1. Note that the design values are also given, for a 3000kg mass with a 2 second natural period, driven by a 0.5$m/s^2$ acceleration, with a design displacement of 5cm. For these values, the four indices discussed above are \{$I_c, I_f, I_v, I_m$\} = \{8.99, 0.717, 51.8, 130\}. 

\[25\]
Table 2.1: Example design parameters for the EC3 ballscrew and AKM42E motor from Danaher Motion.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_e$</td>
<td>0.77N-m/A</td>
<td>$B$</td>
<td>1.2×10^{-4}N-m-s</td>
</tr>
<tr>
<td>$R_c$</td>
<td>2.41Ω</td>
<td>$c_{e,short}$</td>
<td>85130N-s/m</td>
</tr>
<tr>
<td>$L_c$</td>
<td>8.93×10^{-3}H</td>
<td>$f_{rat}$</td>
<td>1076N</td>
</tr>
<tr>
<td>$l$</td>
<td>2.55×10^{-3}m/rad</td>
<td>$m$</td>
<td>3000kg</td>
</tr>
<tr>
<td>$f_{screw}$</td>
<td>7.2kN</td>
<td>$k$</td>
<td>3×10^4N/m</td>
</tr>
<tr>
<td>$T_{rat}$</td>
<td>2.74N-m</td>
<td>$\dot{x}_0$</td>
<td>0.16m/s</td>
</tr>
<tr>
<td>$v_d$</td>
<td>1.4V</td>
<td>$A_0$</td>
<td>0.5m/s^2</td>
</tr>
<tr>
<td>$J$</td>
<td>1.5×10^{-4}N-m-s^2</td>
<td>$\omega_0$</td>
<td>0.5Hz</td>
</tr>
</tbody>
</table>

2.3 Experimental Characterization

2.3.1 Mechanical Model Formulation

To illustrate the characteristics of the electromagnetic transducer described in the previous section, experiments were conducted in which the transducer displacement was controlled via a hydraulic actuator, and various response quantities were measured. As previously discussed, it is known from the manufacturing specifications that the ballscrew itself has some amount of inertia and viscous damping. In addition, sinusoidal oscillations are present in the force-displacement plane and are due to the sliding friction force of the ballscrew. In most applications of precision ballscrews (which are for the purpose of positioning) this force is compensated for, via feedback, and oftentimes its characteristics are not extensively tested by manufacturers. Furthermore, hysteresis in the force-velocity plane can be seen at low velocities (i.e., as the motion of the ballscrew changes direction). To formulate a mechanical model that effectively captures these nonlinear effects, we compare the fit responses of two different models to experimental responses.

Consider the mechanical models of the electromagnetic transducer in Figure 2.6. The Bouc-Wen model in Figure 2.6(a) includes a Bouc-Wen force element (Wen, 1976) to model the hysteresis in the force-velocity plane. Hysteresis effects have
been extensively studied in the literature and a common model used to describe this phenomenon is the Bouc-Wen model, which can exhibit a wide variety of hysteretic behavior. Specifically, the seminal study by Spencer et al. (1997) includes a Bouc-Wen force element to model the hysteresis that is present in their experimental MR damper. However, we have found that the Bouc-Wen model is not able to accurately model the hysteretic effects over a broad range of frequencies, especially when its parameters are fit for experimental data at a single frequency. As such, we propose the mechanical model in Figure 2.6(b), which is analytically tractable and can accurately model the hysteretic effects over wide range of frequencies with a nonlinear spring. We show in Section 2.3.3 that the proposed model outperforms the Bouc-Wen model for a random displacement test.

We begin by deriving the equations of motion that describe the models in Figure 2.6. The inertia that is present in the ballscrew is represented by a rack-and-pinion (i.e., inerter) element (Smith, 2002) with equivalent mass $m_d$. Similarly, the viscous damping that is present in the ballscrew is represented by $c_d$. In addition, we include the linear spring $k_d$ to account for the stiffness that is present in the device. The bearing friction of ballscrew is modeled by a summation of two sinusoids that are a
function of displacement; i.e.,

\[ f_{\text{bearings}} = (\gamma_1 \sin (\lambda_1 x + \phi_1) + \gamma_1 \sin (\lambda_1 x + \phi_1)) \text{sgn}(\dot{x}) \]  

(2.33)

where \( \lambda_1 = 1/l \) and \( \lambda_2 = 2\pi/r \). The distance \( r \) is the distance in the \( x \)-direction that the transducer travels, to cause a bearing to roll the length of its diameter in the track of the ballscrew (assuming a no-slip condition between the bearings and the track); i.e.,

\[ r = 2d \sin(\beta) \]  

(2.34)

where \( d \) is equal to the diameter of one bearing (i.e., \( d = 0.29 \text{cm} \)) and \( \beta \) is equal to the helical angle of the ballscrew (i.e., \( \beta = 17 \text{ degrees} \)).

For the Bouc-Wen model in Figure 2.6(a), we define the Bouc-Wen force \( f_{\text{bouc}} \) as

\[ f_{\text{bouc}} = \alpha_b z \]  

(2.35)

where the internal degree-of-freedom \( z \) evolves according to the differential equation

\[ \dot{z} = (A_b - |z|^n (a_b \text{sgn}(\dot{x}z) + (A_b - a_b))) \dot{x} . \]  

(2.36)

It should be noted that the parameter \( n \) can be any positive integer, but we have set \( n = 2 \) for this model. The remaining parameters (i.e., \( \alpha_b, A_b, \) and \( a_b \)) can be adjusted to produce a wide range of hysteretic forms in the force-velocity plane. Thus, we can solve for the total device force \( f \) by equating the forces on either side of the rigid bar; i.e.,

\[ f = m_d \ddot{x} + (c_d + c_e) \dot{x} + k_d x + f_{\text{bouc}} + f_{\text{bearings}} . \]  

(2.37)

For the proposed model in Figure 2.6(b), we note that the internal degree-of-freedom represented by \( y \) is related to the tandem mounting of the synchronous machine and the ballscrew via the timing belt. Because the timing belt has some elasticity, we can model this effect by a nonlinear spring; i.e.,

\[ f_{\text{belt}} = k_{b1} y + k_{b3} y^3 \]  

(2.38)
where $k_{b1}$ and $k_{b3}$ are spring constants. An additional viscous damper $c_b$ is added to this degree-of-freedom to account for viscous damping associated with the shaft of the motor. To determine the analytical expression for the total device force $f$, we start by balancing the forces about the rigid bar associated with the $y$ degree-of-freedom; i.e.,

$$f_{belt} + c_b \dot{y} = m_d (\ddot{x} - \ddot{y}) + c_d (\dot{x} - \dot{y}) + k_d (x - y) + f_c \text{sgn} (\dot{x} - \dot{y}).$$

(2.39)

Next, solving Equation (2.39) for $\ddot{y}$ results in

$$\ddot{y} = \frac{1}{m_d} (m_d \ddot{x} + c_d (\dot{x} - \dot{y}) + k_d (x - y) + f_c \text{sgn} (\dot{x} - \dot{y}) - f_{belt} - c_b \dot{y}).$$

(2.40)

where $y$ and $\dot{y}$ can be found through numerical integration with the measurements $x$, $\dot{x}$, and $\ddot{x}$ treated as inputs to the differential equation. The total force generated by the system can be found by summing the forces in the upper and lower sections of the system in Figure 2.6(b). Thus, the total force can be expressed by

$$f = m_d (\ddot{x} - \ddot{y}) + c_d (\dot{x} - \dot{y}) + k_d (x - y) + f_c \text{sgn} (\dot{x} - \dot{y}) + f_{bearings} + c_e \dot{x},$$

(2.41a)

$$= f_{belt} + c_b \dot{y} + f_{bearings} + c_e \dot{x}. \quad (2.41b)$$

### 2.3.2 Error Quantification

The parameters in the models in Figure 2.6 are determined using a Levenberg-Marquardt nonlinear least squares algorithm (Levenberg, 1944; Marquardt, 1963). Levenberg-Marquardt methods are appealing because they use both gradient-descent and Gauss-Newton methods to minimize a least squares objective, which results in faster convergence times. To evaluate the statistical accuracy of the fit parameters, we compute their asymptotic standard error. Standard parameter errors can be used to determine confidence intervals for parameter values. First, we define the vector $\mathbf{f} \in \mathbb{R}^m$ as the experimentally measured force, the vector $\hat{\mathbf{f}} \in \mathbb{R}^m$ as the predicted force, and $\mathbf{p} \in \mathbb{R}^n$ as the vector of fit parameters (in this case $n = 10$ for the
Bouc-Wen model and \( n = 11 \) for the proposed model). Therefore, we have that the parameter covariance matrix can be represented by

\[
\Sigma_p = (J_p^T W J_p)^{-1}
\] (2.42)

where \( J_p \in \mathbb{R}^{m \times n} \) is the Jacobian of the predicted force with respect to each parameter (i.e., \( J_p = \partial \tilde{f} / \partial p \)) and \( W \in \mathbb{R}^{m \times m} \) is a diagonal weighting matrix where \( W_{ii} = 1/w_i^2 \). It should be noted that \( w_i \) is equal to the mean square measurement error; i.e.,

\[
w_i^2 = \frac{1}{m - n + 1} (f - \hat{f})^T (f - \hat{f}), \quad \forall i .
\] (2.43)

The asymptotic standard error of parameter \( p_i \) is just

\[
\sigma_p = \sqrt{(\Sigma_p)_{ii}}.
\] (2.44)

It is convenient to represent these standard errors as percentages. As such, we define the percent standard error of each fit parameter \( p_i \) as

\[
\epsilon_p = \frac{\sigma_p}{p_i} \times 100.
\] (2.45)

In addition quantifying the statistical accuracy of the fitted parameters, we will also quantify the errors between the experimentally measured force and the predicted force using the error norms defined in Spencer et al. (1997). In this study, error norms for time, displacement, and velocity can be calculated by the following three expressions

\[
E_t = \frac{\sqrt{\sum_{i=1}^{m} (f_i - \hat{f}_i)^2}}{\sqrt{\sum_{i=1}^{m} (f_i - \mu_f)^2}}, \quad E_x = \frac{\sqrt{\sum_{i=1}^{m} (f_i - \hat{f}_i)^2 |\dot{x}_i|}}{\sqrt{\sum_{i=1}^{m} (f_i - \mu_f)^2}}, \quad E_{\ddot{x}} = \frac{\sqrt{\sum_{i=1}^{m} (f_i - \hat{f}_i)^2 |\ddot{x}_i|}}{\sqrt{\sum_{i=1}^{m} (f_i - \mu_f)^2}}
\] (2.46)

where \( \mu_f \) is the mean value of the measured force, \( \dot{x} \in \mathbb{R}^m \) is a vector of measured velocity, and \( \ddot{x} \in \mathbb{R}^m \) is a vector of measured acceleration.
2.3.3 Parameter Fit and Model Verification

To determine the fit for the parameters in the models in Figure 2.6, we start by following the procedure outlined in the study by Spencer et al. (1997). In that study, the parameters for the mechanical model are determined by controlling the displacement of the device using a sinusoidal displacement for two full oscillations. To test this procedure, we use a 0.5Hz sinusoidal displacement with an amplitude equal to 7.5cm. A plot of this displacement input can be seen in Figure 2.7. It should be noted that the terminals of the motor are left open for this test (i.e., $c_e = 0$). During this test, force, displacement, velocity, and acceleration data was collected using a dSpace DS1103 data acquisition system at a sample rate of 1kHz. Next, the mea-

![Figure 2.7: Displacement input applied to the device in the sine test.](image)

Table 2.2: Parameter values and asymptotic standard errors for the Bouc-Wen model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>$\sigma_p$</th>
<th>$\epsilon_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_d$</td>
<td>$8.64 \times 10^{-4}$ kg</td>
<td>$2.61 \times 10^{-5}$ kg</td>
<td>3.02%</td>
</tr>
<tr>
<td>$c_d$</td>
<td>669 N-s/m</td>
<td>0.921 N-s/m</td>
<td>0.14%</td>
</tr>
<tr>
<td>$k_d$</td>
<td>255 N/m</td>
<td>0.758 N/m</td>
<td>0.30%</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>11.4</td>
<td>0.406</td>
<td>3.56%</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>2.50</td>
<td>0.0372</td>
<td>1.49%</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>10.5</td>
<td>0.408</td>
<td>3.87%</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>0.312</td>
<td>0.0207</td>
<td>6.63%</td>
</tr>
<tr>
<td>$\alpha_b$</td>
<td>110 N/m</td>
<td>0.590 N/m</td>
<td>0.54%</td>
</tr>
<tr>
<td>$A_b$</td>
<td>23.6</td>
<td>0.658</td>
<td>2.97%</td>
</tr>
<tr>
<td>$a_b$</td>
<td>51.0</td>
<td>3.03</td>
<td>5.94%</td>
</tr>
</tbody>
</table>
Figure 2.8: Comparison between experimentally measured responses and the Bouc-Wen model fit to this data: (a) force versus time; (b) force versus displacement; (c) force versus velocity.

Measured data was used as inputs for the Bouc-Wen model device force $f$ (i.e., Equation (2.37)) and the Levenberg-Marquardt method was implemented to solve for the optimal parameters in the model. Optimal fit values for the parameters as well as the asymptotic standard error percentages for each parameter are listed in Table 2.2. The fit responses using the optimal parameters are compared to measured responses in Figure 2.8.

From the fourth column in Table 2.2, we see that the parameters associated with the high-frequency force oscillations attributable to the ballscrew mechanics are fit with the least level of confidence. However, the parameters associated with the viscous damping, stiffness, and amplitude of the Bouc-Wen hysteresis force are fit with the greatest level of certainty. For this parameter fit test, the error norms given
in Equation (2.46) were calculated to be $E_t = 0.142$, $E_x = 0.258$, and $E_\dot{x} = 0.442$.

To verify the accuracy of the Bouc-Wen model when the terminals of the motor are left open, the device was excited with an 18 second random displacement. The random displacement is characterized by bandpass filtered white noise. The particular filter used to generate the random displacement is a second-order filter where we have set the center of passband equal to 0.5Hz with an acceleration quality factor of 0.5. Figure 2.9 illustrates the random displacement used for this test. Using the parameters fit for the sine test (i.e., column 2 of Table 2.2), we compare the predicted response of the device using the Bouc-Wen model to experimental data. The plots in Figure 2.10 show that the Bouc-Wen model does a poor job of predicting the response of the device to a random displacement. The norms given in Equation (2.46) were calculated to be $E_x = 0.312$, $E_\dot{x} = 0.459$, and $E_\ddot{x} = 1.35$ for this test.

Motivated by the poor way in which the fit parameters for the Bouc-Wen model perform during a random displacement test, we fit the proposed model by backdriving the device over a range of frequencies. A sine-sweep displacement ranging from 0.2Hz to 2Hz with a velocity envelope ranging from 4.5cm/s to 7.5cm/s was generated over 44 seconds and was used for this test. A plot of this displacement input can be seen in Figure 2.11. Again, it should be noted that the terminals of the motor were left open (i.e., $c_e = 0$) and that the same response quantities were measured at 1kHz with the dSpace DS1103 data acquisition system. The measured data was used as inputs for the proposed model device force $f$ (i.e., Equation (2.41)) and the Levenberg-Marquardt method was implemented to solve for the optimal parameters in the model. Optimal fit values for the parameters as well as the asymptotic standard error percentages for each parameter are listed in Table 2.3. The fit responses using the optimal parameters are compared to measured responses in Figure 2.12.

From the fourth column of Table 2.3, we see that the parameters associated with the high-frequency force oscillations attributable to the ballscrew mechanics
**Figure 2.9:** Displacement input applied to the device for the random displacement test.

**Figure 2.10:** Comparison between predicted and experimentally obtained responses using the Bouc-Wen model for the random displacement test: (a) force versus time; (b) force versus displacement; (c) force versus velocity.
are fit with the least level of confidence. However, the parameters associated with the nonlinear belt mechanics, the Coulomb friction force, and the viscous damping are fit with the greatest level of certainty. Both of these observations are consistent with sinusoidal test for the Bouc-Wen model (with the exception the nonlinear belt mechanics). The error norms given in Equation (2.46) were calculated to be $E_t = 0.209$, $E_x = 0.325$, and $E_{\dot{x}} = 0.919$ for this test. The error norms for time and displacement are comparable with the error norms for the Bouc-Wen model, but the velocity error for this model is approximately twice as large as the velocity error for the Bouc-Wen model.

To verify the accuracy of the proposed model when the terminals of the motor

![Figure 2.11: Displacement input applied to the device in the sine-sweep test.](image)

**Table 2.3:** Parameter values and asymptotic standard errors for the proposed model.

<table>
<thead>
<tr>
<th>Parameter $</th>
<th>Value $</th>
<th>$\sigma_p $</th>
<th>$\epsilon_p $</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_d$</td>
<td>$2.26\times10^{-5}$kg</td>
<td>$2.64\times10^{-8}$kg</td>
<td>0.12%</td>
</tr>
<tr>
<td>$c_d$</td>
<td>575N-s/m</td>
<td>0.339N-s/m</td>
<td>0.06%</td>
</tr>
<tr>
<td>$k_d$</td>
<td>627N/m</td>
<td>0.425N/m</td>
<td>0.07%</td>
</tr>
<tr>
<td>$f_c$</td>
<td>143.1N</td>
<td>0.0407N</td>
<td>0.03%</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>6.53</td>
<td>0.245</td>
<td>3.75%</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>1.97</td>
<td>0.0383</td>
<td>1.94%</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>11.4</td>
<td>0.246</td>
<td>2.15%</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>0.974</td>
<td>0.0217</td>
<td>2.23%</td>
</tr>
<tr>
<td>$k_{b1}$</td>
<td>114N/m</td>
<td>0.252N/m</td>
<td>0.22%</td>
</tr>
<tr>
<td>$k_{b3}$</td>
<td>$1.54\times10^9$N/m$^3$</td>
<td>$6.88\times10^6$N/m$^3$</td>
<td>0.04%</td>
</tr>
<tr>
<td>$c_b$</td>
<td>$1.60\times10^4$N-s/m</td>
<td>5.14N-s/m</td>
<td>0.03%</td>
</tr>
</tbody>
</table>
are left open, the device was excited with an 18 second random displacement. The random displacement used for this test is the same as the random displacement used for the Bouc-Wen model and a plot can be seen in Figure 2.9. Using the parameters fit for the sine-sweep test (i.e., column 2 of Table 2.3), we compare the predicted response of the device using the proposed model to experimental data. The plots in Figure 2.13 show that the model accurately predicts the behavior of the device. For this test, the error norms given in Equation (2.46) were determined to be $E_t = 0.177$, $E_x = 0.231$, and $E_{\dot{x}} = 0.618$. All three of these error norms are approximately 50% of the value of the error norms calculated for the Bouc-Wen model. Thus, we can conclude that the proposed model with the parameters fit using a sine-sweep displacement can be used to accurately predict the response of
Figure 2.13: Comparison between predicted and experimentally obtained responses using the proposed model for the random displacement test: (a) force versus time; (b) force versus displacement; (c) force versus velocity.

the device. As such, we will use the proposed model for the remainder of the tests in this chapter. Furthermore, the proposed model has the added benefit that all of its force elements are physically motivated, which is not the case for the Bouc-Wen force element.

2.3.4 Response to a Fluctuating Electromechanical Force

The response of the electromagnetic transducer to all of the previous tests have been for the case where the terminals of the motor have been left open. However, the performance of an energy harvesting system that utilizes this device will depend on the response of the device with electronics controlling the rate power extraction. As such, we must show that the proposed model can accurately predict the behavior of
the device when the electronics introduce an additional fluctuating electromechanical force. As previously mentioned, one of the simplest ways to harvest power from an electromagnetic transducer is to have the electronics simulate a resistive load across the terminals of the transducer. In this case, we have that the additional electromechanical force is the force from Equation (2.24). Three tests are presented to demonstrate the accuracy of the proposed model with a fluctuating electromechanical force, including (1) step response; (2) constant $R_L$, random displacement; and (3) random $R_L$, random displacement.

The step response test consisted of applying a triangular displacement with a constant velocity to the device and with a step change in the load resistance as the device passes through the middle of the stroke. A triangle wave with an amplitude of 5.25cm at 0.25Hz was used for this test and can be seen in Figure 2.14. The plots presented in Figure 2.15 show the response of the device to this test. At the beginning of the test, the servo drive is used to simulate the device in open circuit mode. During the mid-way point of the second oscillation, a commanded load resistance of 10Ω is applied to the servo drive, which results in the device reaching its rated force in 6ms. This rise time is comparable to a MR damper of similar size and force rating (Spencer et al., 1997). The error norms given in Equation (2.46) were calculated to be $E_t = 0.132$, $E_x = 0.234$, and $E_v = 0.497$ for this test.

![Figure 2.14: Displacement input applied to the device for the step response test.](image-url)
Figure 2.15: Comparison between predicted and experimentally obtained responses for the step response test: (a) force versus time; (b) force versus displacement; (c) force versus velocity.

Figure 2.16: Close-up view of the force versus displacement for the step response test.
During the step response test, we observe some amount of overshoot and ringing in the device force after the step change in load resistance is applied to the servo drive. The close-up view of the force response during the step change in resistance can be seen in Figure 2.16. The oscillations occurring before the force measurement reaches a steady state is a result of the dynamic interaction between servo drive and the finite coil inductance of the motor. Because internal current tracking of the servo drive is accomplished by proportional control, a step in the load resistance causes a ringing effect in the force measurement for a brief period of time until the coils become fully magnetized. We see that the predicted force response in Figure 2.16 does not capture the oscillations. This is due to the fact that the oscillations are a result of the dynamics of the servo drive, which is not included in the proposed model. It would be possible to design a compensator using feedback to reduce these oscillations, but this is beyond the scope of this thesis.

In the second test conducted to verify the model, the device was excited with the same random displacement as in Figure 2.9, but the servo drive is used to simulate a constant load resistance across the terminals of the motor. In this case a 16Ω load resistance was used to ensure that the device would not exceed its rated force limit. The response of the device to this test can be seen in Figure 2.17. As seen here, the model accurately predicts the behavior of the device. The error norms given in Equation (2.46) were calculated to be $E_t = 0.115$, $E_\delta = 0.203$, and $E_\dot{x} = 0.378$.

For the final verification test, the load resistance simulated by the servo drive was chosen to randomly saturate between two values. The upper bound on the load resistance is 800Ω and the lower bound on the load resistance is 16Ω. These values were chosen so that the electromechanical force of the device would fluctuate between the force resulting from open circuit mode and the rated force. A plot of the random load resistance can be seen in Figure 2.18. Again, the random displacement that is applied to the device is the same as previous test. The results presented
Figure 2.17: Comparison between predicted and experimentally obtained responses for the constant $R_L$, random displacement test: (a) force versus time; (b) force versus displacement; (c) force versus velocity.

In Figure 2.19 confirm that we have excellent agreement between the experimental response and the model. For this test, the error norms given in Equation (2.46) were determined to be $E_t = 0.138$, $E_x = 0.237$, and $E_v = 0.484$. 
Figure 2.18: Applied $R_L$ command to the device in the random displacement, random resistance test.

Figure 2.19: Comparison between predicted and experimentally obtained responses for the random $R_L$, random displacement test: (a) force versus time; (b) force versus displacement; (c) force versus velocity.
The focus of this chapter is on the derivation of the optimal, unconstrained energy harvesting controller from a stochastic disturbance. This controller is based on the assumptions that the power electronics are capable of two-way power flow and that the parasitic power losses in the electronics are resistive. In order to formulate the optimal energy harvesting controller, we approximate the disturbance and harvester dynamics as finite-dimensional state space systems. We begin by representing the disturbance dynamics by a second-order band-pass filter that maps a white noise process into the disturbance acceleration that excites the energy harvester. Next, the dynamics of the harvester are formulated as a self-dual state space. From these state space models, we derive the augmented disturbance and harvester model.

The optimal state-feedback control law is derived using LQG control theory where current imposed by the electronics is treated as the control input. We also derive the optimal static admittance for the augmented model. The optimal static and dynamic admittances are realized for a simple energy harvesting example consisting of a base excited SDOF oscillator with electromagnetic coupling. Finally, we show the improvement in terms of the average power generated that can be achieved through
the implementation of the optimal dynamic admittance.

To draw the connection between the modeling of the transducer presented in Chapter 2 with the energy harvesting theory presented in this chapter, we make the following three assumptions. The first assumption is that we ignore the nonlinear mechanical dynamics present in the transducer. In Chapter 5, we account for these nonlinearities in the energy harvesting theory using statistical linearization. The second assumption is that the electrical dynamics of the transducer are controlled such that only the quadrature current is non-zero. Under this assumption, the control current will be expressed by $i$. As such, the theory presented in this chapter is valid for three-phase as well as single-phase transducers. Finally, the third assumption is that the output voltage across the terminals of the transducer is equivalent to the quadrature back emf of the three-phase transducer. In order to simplify the notation, this voltage by be denoted by $v$.

3.1 State-Space Modeling

3.1.1 Disturbance Model

Let $a(t)$ be the acceleration of the disturbance that excites the energy harvesting system. We assume that $a(t)$ is characterized as filtered noise which has a power spectral density equal to

$$
\Phi_a(\omega) = \left| \frac{q j \omega}{-\omega^2 + 2 \omega_a \zeta_a j \omega + \omega_a^2} \right|^2
$$

(3.1)

where $\omega_a$ is the center of the passband of $a(t)$, and $\zeta_a$ determines the spread of its frequency content. This particular power spectral density is equivalent to a second-order band-pass filter. For such a process, it is straight-forward to represent the
disturbance dynamics by a two-dimensional state space of the form

\[
\frac{d}{dt}x_a(t) = A_a x_a(t) + B_a w(t) \quad (3.2a)
\]

\[
a(t) = C_a x_a(t) \quad (3.2b)
\]

where \(w(t)\) is a white noise process with spectral intensity equal to unity. The parameter \(q\) in (3.1) is adjusted such that irrespective of \(\omega_a\) and \(\zeta_a\), \(a(t)\) has a consistent standard deviation of \(\sigma_a\); i.e.,

\[
\sigma_a = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_a(\omega) d\omega} \quad (3.3)
\]

This allows us to compare disturbances of varying spectral content, but equal intensity. We refer to the “narrowband limit” for the disturbance model as the case in which \(\zeta_a \to 0\). Similarly, refer to “broadband limit” as the case in which \(\zeta_a \to \infty\).

The value of \(q\) that satisfies Equation (3.3) can be found by solving a Lyapunov equation for the stationary covariance of the disturbance states. If we define the disturbance dynamics, input, and output matrices as

\[
A_a = \begin{bmatrix} 0 & 1 \\ -\omega_a^2 & -2\zeta_a \omega_a \end{bmatrix},
\]

\[
B_a = \begin{bmatrix} 0 \\ q \end{bmatrix}^T,
\]

\[
C_a = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

and the stationary covariance matrix of the disturbance states as \(\Sigma_a = \mathcal{E}\{x_a(t)x_a^T(t)\}\), then the Lyapunov equation that is satisfied in stationarity is

\[
A_a \Sigma_a + \Sigma_a A_a^T + B_a B_a^T = 0 \quad (3.7)
\]

It turns out that the stationary covariance matrix \(\Sigma_a\) can be defined as

\[
\Sigma_a = \begin{bmatrix} \sigma_a^2 & 0 \\ 0 & \sigma_a^2 \end{bmatrix} \quad (3.8)
\]
Since we know that $\sigma_a^2 = C_a \Sigma_a C_a^T$, we can solve Equation (3.7) for $q$ in terms of the known parameters. In this case we have that

$$q = 2\sigma_a \sqrt{\zeta_a} \omega_a .$$  \hspace{1cm} (3.9)

A plot illustrating $\Phi_a(\omega)$ for various values of $\zeta_a$ can be seen in Figure 3.1. It should be noted that we have set $\omega_a = 0.5\text{Hz}$ and $\sigma_a = 9.81\text{m/s}^2$, which are reasonable values for a multi-story building or long-span bridge vibrating at its fundamental mode.

### 3.1.2 Harvester Model

We assume the harvester is a dissipative system and that the harvester dynamics can be approximated by a finite-dimensional linear state space. Let $H_a(s)$ and $H_i(s)$ be the transfer functions from the disturbance acceleration $a(t)$ and the control current $i(t)$ to the voltage generated by the system $v(t)$, respectively, where $s$ denotes the Laplace variable. We will assume both $H_a(s)$ and $H_i(s)$ to be strictly proper. The condition that $H_a(s)$ is strictly proper ensures a finite bandwidth of the response of the harvester. It is pointed out in Scruggs (2010) that problems may arise if $H_i(s)$ is merely proper, but not strictly proper. This case will arise if the transducer has
a non-zero terminal resistance, which can be corrected by re-defining \( v(t) \) and the electronic losses \( R \). In addition, we assume that \( H_i(s) \) is weakly strictly positive real (WSPR) (Brogliato et al., 2007). Since \( H_i(s) \) can be thought of as the driving point impedance of the harvester as seen at the terminals of the transducer, this implies that \( H_i(s) \) can always be realized by an asymptotically-stable circuit with passive components (i.e., resistors, capacitors, and inductors).

With these assumptions, there always exists a self-dual state space realization (Lozano-Leal and Joshi, 1988) for the harvester, of the form

\[
\frac{d}{dt}x_h(t) = A_h x_h(t) + B_h i(t) + G_h a(t) \quad (3.10a)
\]

\[
v(t) = B_h^T x_h(t) . \quad (3.10b)
\]

The state-space representation of the harvester implies that the transfer functions from current \( i(t) \) and acceleration \( a(t) \) to voltage \( v(t) \) can be represented, respectively, as

\[
H_i(s) = B_i^T (sI - A_h)^{-1} B_i \quad (3.11)
\]

\[
H_a(s) = B_i^T (sI - A_h)^{-1} B_a . \quad (3.12)
\]

In the above realization, the total dissipation in the harvester at time \( t \) is \(-\frac{1}{2}x_h(t)^T (A_h + A_h^T)x_h(t) \geq 0\), and the total energy stored in the harvester is \(\frac{1}{2}x_h(t)^T x_h(t)\). The WSPR assumption allows us to assume that the pair \((A_h, A_h + A_h^T)\) is observable, which implies that no free response of the harvester can exhibit zero internal dissipation over any finite interval.

3.1.3 Augmented Model

It is straightforward to combine the disturbance and harvester dynamics into an augmented state-space system; i.e.,

\[
\frac{d}{dt}x(t) = Ax(t) + Bi(t) + Gw(t) \quad (3.13a)
\]

\[
v(t) = B^T x(t) \quad (3.13b)
\]
where the augmented matrices $A$, $B$, and $G$ are

$$A = \begin{bmatrix} A_h & G_h C_a \\ 0 & A_a \end{bmatrix},$$

(3.14)

$$B = \begin{bmatrix} B^T_h & 0 \end{bmatrix}^T,$$

(3.15)

$$G = \begin{bmatrix} 0 & B^T_a \end{bmatrix}^T.$$  

(3.16)

and the resultant augmented state vector is

$$x(t) = \begin{bmatrix} x_f^T(t) & x_u^T(t) \end{bmatrix}^T.$$  

(3.17)

We make the assumption that $(A, B^T)$ is observable and $(A, [B \ G])$ is controllable. If this is not the case for the original system model, we assume that the dimension of the augmented state space has been reduced to a minimal realization. The augmented realization gives rise to a new transfer function $H_w(s)$, which is from white noise $w(t)$ to voltage $v(t)$, and can be written as

$$H_w(s) = B^T(sI - A)^{-1}G.$$  

(3.18)

### 3.2 SDOF Example

To illustrate the properties described in the previous section, we introduce a simple energy harvesting example. Consider the SDOF resonant oscillator with mass $m$, damping $b$, and stiffness $k$ in Figure 3.2. A transducer is attached between the base and the moving mass such that $f_e(t) = c_e i(t)$ where $c_e$ is the coupling coefficient. This system corresponds to an energy harvester with electromagnetic coupling. For the remainder of the this chapter we will neglect any additional electrical and mechanical dynamics that are present in the transducer such as the inductance of the coil. Similar electromagnetic energy harvesting systems have been demonstrated in Ward and Behrens (2008); Glynne-Jones et al. (2004); Scruggs and Behrens (2011).
The governing equations for this system are

\[ m\ddot{r}(t) + b\dot{r}(t) + kr(t) = ma(t) + c_i(t) \]  \hspace{1cm} (3.19a)

\[ v(t) = c_\epsilon \dot{r}(t) \]  \hspace{1cm} (3.19b)

where \( r(t) \) is the relative displacement of the harvester. The corresponding power spectral density for the disturbance acceleration \( a(t) \) is defined in Equation (3.1).

If we make the following substitutions for time and relative displacement, respectively,

\[ t = \sqrt{\frac{m}{k}} \tau, \]  \hspace{1cm} (3.20)

\[ r(t) = \left( \frac{m\sigma_a}{k} \right) \tilde{r}(\tau) \]  \hspace{1cm} (3.21)

then the harvester dynamics can be expressed in nondimensional coordinates; i.e.,

\[ \ddot{\tilde{r}}(\tau) + d\dot{\tilde{r}}(\tau) + \tilde{r}(\tau) = \ddot{\tilde{a}}(\tau) + \ddot{\tilde{i}}(\tau) \]  \hspace{1cm} (3.22a)

\[ \tilde{v}(\tau) = \dot{\tilde{r}}(\tau) \]  \hspace{1cm} (3.22b)

where the nondimensional damping factor is

\[ d = \frac{b}{\sqrt{mk}}. \]  \hspace{1cm} (3.23)
It follows that the nondimensional acceleration, current, and voltage are, respectively,

\[ a(t) = \sigma_a \bar{a}(\tau), \quad (3.24) \]

\[ i(t) = \left( \frac{m \sigma_a}{c_e} \right) \bar{i}(\tau), \quad (3.25) \]

\[ v(t) = \left( c_e \sigma_a \sqrt{\frac{m}{k}} \right) \bar{v}(\tau). \quad (3.26) \]

The nondimensionalized harvester dynamics can be modeled by a two-dimensional state vector \( \mathbf{x}_h(\tau) = \left[ \bar{r}(\tau) \; \dot{\bar{r}}(\tau) \right]^T \) such that

\[ \mathbf{A}_h = \begin{bmatrix} 0 & 1 \\ -1 & -d \end{bmatrix}, \quad (3.27) \]

\[ \mathbf{B}_h = \begin{bmatrix} 0 & 1 \end{bmatrix}^T, \quad (3.28) \]

\[ \mathbf{G}_h = \begin{bmatrix} 0 & 1 \end{bmatrix}^T. \quad (3.29) \]

We assume that the harvester has been tuned such that its natural frequency is in the center of the disturbance passband. This implies that the power spectrum of \( \bar{a}(\tau) \) can be expressed (in normalized frequency \( \bar{\omega} = \omega \sqrt{m/k} \)) as

\[ \Phi_{\bar{a}}(\bar{\omega}) = \left| \frac{\bar{q} \bar{\omega}}{-\bar{\omega}^2 + 2\zeta_a \bar{\omega} + 1} \right|^2, \quad (3.30) \]

with \( \bar{q} \) chosen such that \( \int_{-\infty}^{\infty} \Phi_{\bar{a}}(\bar{\omega}) \, d\bar{\omega} = 1 \). It turns out that the value of \( \bar{q} \) which brings this about is \( \bar{q} = 2\sqrt{\zeta_a} \). Thus, the nondimensionalized disturbance dynamics can be modeled by a two-dimensional state vector \( \mathbf{x}_a(\tau) = \left[ \int \bar{a}(\tau) \; \bar{a}(\tau) \right]^T \) such that

\[ \mathbf{A}_a = \begin{bmatrix} 0 & 1 \\ -1 & -2\zeta_a \end{bmatrix}, \quad (3.31) \]

\[ \mathbf{B}_a = \begin{bmatrix} 0 & 2\sqrt{\zeta_a} \end{bmatrix}^T, \quad (3.32) \]

\[ \mathbf{C}_a = \begin{bmatrix} 0 & 1 \end{bmatrix}. \quad (3.33) \]
Combining the harvester and disturbance models into an equivalent augmented form results in the following representation for the dynamics and input matrices

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & -d & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & -2\zeta_a
\end{bmatrix}, \quad (3.34)
\]

\[
B = \begin{bmatrix}
0 & 1 & 0 & 0
\end{bmatrix}^T, \quad (3.35)
\]

\[
G = \begin{bmatrix}
0 & 0 & 0 & 2\sqrt{\zeta_a}
\end{bmatrix}^T. \quad (3.36)
\]

We will henceforth uniformly assume that the electromagnetic example has been nondimensionalized as described above. To ease the notation, we will do away with all overbars and refer to the nondimensional time as \( t \).

### 3.3 Optimal State-Feedback Control Law

For the moment, we presume that the entire system state, \( x(t) \) is available for feedback. In this case, the design problem involves the design of a causal feedback law \( x(t) \mapsto i(t) \). The objective is to find the particular feedback law which maximizes the average power generated in stationary stochastic response. To do this, first we define the power delivered to storage as the power extracted from the transducer, minus the transmission losses in the power electronic circuitry. If we approximate these losses as resistive, with some resistance \( R \), then the power delivered to storage is

\[
P_s(t) = -i(t)v(t) - Ri^2(t). \quad (3.37)
\]

If we take the expectation of both sides of Equation (3.37), then we can define the average power generated as the expectation of the power delivered to storage; i.e.,

\[
P_{gen} = -\mathcal{E}\{i(t)B^Tx(t) + Ri^2(t)\}. \quad (3.38)
\]
This is equivalent to a LQG optimal control problem. We define the cost function as

$$J = \mathcal{E}\left\{ \begin{bmatrix} x(t) \\ i(t) \end{bmatrix}^T \begin{bmatrix} 0 & \frac{1}{2}B \\ \frac{1}{2}B^T & \frac{1}{2}R \end{bmatrix} \begin{bmatrix} x(t) \\ i(t) \end{bmatrix} \right\}. \quad (3.39)$$

Minimizing the cost function $J$ is equivalent to maximizing $\bar{P}_{\text{gen}}$. Note that the performance functional is sign-indefinite. This reflects the observation that if $i(t)$ is controlled poorly then the efficiency of conversion can be negative, implying drainage of energy from storage rather than accumulation.

We make the assumption that the H-bridge can control the current $i(t)$ with high enough bandwidth such that its dynamics lie outside the frequency band of the disturbance. This assumption may be unjustified in some applications, but for applications with response bandwidths below about 1kHz, it is often reasonable. Given this assumption, we may view $i(t)$ as a control input which can be regulated as desired. If the transducer is three-phase, then the control design would be for the quadrature current with the direct current controlled such that it equals zero. The following theorem then draws the connection between energy harvesting and optimal LQG control.

**THEOREM 1.** Over the space of all causal, continuous feedback functions, the optimal energy harvesting current is characterized by the linear state feedback relationship

$$i(t) = Kx(t) \quad (3.40)$$

where

$$K = -\frac{1}{R}B^T(P + \frac{1}{2}I) \quad (3.41)$$

and $P = P^T < 0$ is the unique, stabilizing solution to the nonstandard Riccati equation

$$A^TP + PA - \frac{1}{R}(P + \frac{1}{2}I)BB^T(P + \frac{1}{2}I) = 0. \quad (3.42)$$
The optimal average power generated is

\[ P_{\text{gen}} = -G^T P G. \]  

(3.43)

**Proof.** We now present a brief overview of the proof for this theorem. It should be noted that a more complete proof can be found in Scruggs (2010). We start by assuming that \( P = P^T \) can be partitioned into a general \( 2 \times 2 \) block matrix such that

\[ P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}. \]  

(3.44)

Next, we can substitute the relationship for \( P \) as well as the relationships for \( A \) and \( B \) in Equations (3.14) and (3.15) into Equation (3.42) to obtain three new equations; i.e.,

\[ A_h^T P_1 + P_1 A_h - \frac{1}{R} (P_1 + \frac{1}{2} I) B_h B_h^T (P_1 + \frac{1}{2} I) = 0 \]

(3.45)

\[ A_h^T P_2 + P_1 G_h C_a + P_2 A_a - \frac{1}{R} (P_1 + \frac{1}{2} I) B_h B_h^T P_2 = 0 \]

(3.46)

\[ A_a^T P_3 + P_3 A_a + C_a^T G_h^T P_2 + P_2^T G_h C_a - \frac{1}{R} P_2^T B_h B_h^T (P_1 + \frac{1}{2} I) = 0. \]

(3.47)

We can also partition \( K = [K_1 \ K_2] \) such that

\[ K_1 = -\frac{1}{R} B_h^T (P_1 + \frac{1}{2} I), \]

(3.48)

\[ K_2 = -\frac{1}{R} B_h^T P_2. \]

(3.49)

Since \( P_{\text{gen}} \) is an indefinite quadratic form, we must show that the solutions to the three equivalent equations in Equations (3.45)–(3.47) exist and that they result in a stabilizing control law (i.e., \( A + B K < 0 \)).

If we define \( \bar{P}_1 \) as

\[ \bar{P}_1 = P_1 + \frac{1}{2} I \]

(3.50)
then Equation (3.45) is equivalent to the standard Riccati equation

$$-\frac{1}{2}[A_h + A_h^T] + A_h^T P_1 + P_1 A_h - \frac{1}{R} P_1 B_h B_h^T P_1 = 0.$$ (3.51)

This equation is guaranteed to have a unique, stabilizing, and positive definite solution if the following three conditions hold: $-\frac{1}{2}[A_h + A_h^T] \succeq 0$, $(A_h, -\frac{1}{2}[A_h + A_h^T])$ is observable, and $A_h$ is asymptotically stable. All three of these conditions are implied by the fact that $H_i(s)$ is WSPR and that the harvester state-space is self-dual. Thus, we have that a unique $P_1$ exists, which is guaranteed to stabilize $A_h + B_h K_1$. As show in Scruggs (2010), the other components of $P$ can be shown to be stabilizing once a stabilizing $P_1$ is found. Finally, it is a standard result that the performance of any causal feedback law is related to the optimal performance through Equation (3.43)

### 3.4 Optimal Admittance

We now consider the implementation of the way in which power can be harvested for the SDOF energy harvesting example. For this section, we assume that the electronics, control, and storage system can be replaced by the relationship $i(s) = -Y(s)v(s)$ where $Y(s)$ is the admittance. A block diagram illustrating this implementation can be seen in Figure 3.3(a). In this figure, the admittance $Y(s)$ gives the command

![Diagram](image-url)

**Figure 3.3:** Implementation of the admittance relationship by the electronics: (a) block diagram of the admittance; (b) SDOF electromagnetic energy harvester.
current $i^*(s)$ as a function of the measured voltage $v(s)$ of the harvester. The H-bridge then tracks $i^*(s)$ and imposes the control current $i(s)$ across the terminals of the transducer. We assume that the H-bridge can track the command current at a high enough bandwidth such that $i^*(s) = i(s)$. An equivalent realization of this assumption for the SDOF example can be seen in Figure 3.3(b). In this section, we begin by deriving the optimal static admittance. Following this derivation we formulate the optimal dynamic admittance by combining the optimal state-feedback control law with a Luenberger observer.

3.4.1 Optimal Static Admittance

One of the simplest ways to absorb power from an energy harvester is by controlling the electronics to emulate a resistive load. In other words, a static admittance $Y_0$, where $i(t) = -Y_0v(t)$ by convention, is imposed across the terminals of the transducer. The static admittance can then be tuned to maximize the average power generated. Given the relationship of the admittance, the closed loop dynamics of the harvester are

$$\frac{d}{dt}x(t) = [A - Y_0BB^T]x(t) + Gw(t). \quad (3.52)$$

The stationary covariance matrix $\Sigma = \mathcal{E}\{x(t)x^T(t)\}$ is found by solving the Lyapunov equation

$$[A - Y_0BB^T] \Sigma + \Sigma [A - Y_0BB^T]^T + GG^T = 0 \quad (3.53)$$

and the resultant upper bound on power generation is

$$\bar{P}_{gen} = (Y_0 - Y_0^2R)B^T\Sigma B. \quad (3.54)$$

Because the system only has one design parameter (i.e., $Y_0$) in this case, the most straight-forward way to optimize $\bar{P}_{gen}$ is via a one-dimensional line search. One can, for example, employ a bisection algorithm to converge rapidly to the optimal $Y_0$, given $A$, $B$, $G$, and $R$. This is done for two levels of damping over the $\{\zeta_a, R\}$ domain.
Figure 3.4: Dependency of the optimal static admittance and resultant optimal power generation on $\zeta_a$ and $R$: (a) $Y_0$ vs. $\{\zeta_a, R\}$ for $d = 0.01$; (b) $Y_0$ vs. $\{\zeta_a, R\}$ for $d = 0.1$; (c) $\log \bar{P}_{gen}$ vs. $\{\zeta_a, R\}$ for $d = 0.01$; (d) $\log \bar{P}_{gen}$ vs. $\{\zeta_a, R\}$ for $d = 0.1$.

In Figure 3.4. It is interesting to note that the highest levels of $\bar{P}_{gen}$ occur as $\zeta_a \rightarrow 0$ as seen in Figure 3.4(c)–(d). This relationship makes physical sense as it is the case that the optimal controller is just a static admittance for an electromagnetic energy harvester when it is excited by a narrowband disturbance at its resonant frequency.

### 3.4.2 Optimal Dynamic Admittance

In general, the optimal state-feedback control law requires knowledge of every state in system. It is often the case that measuring every state is not feasible. As such, it has become common practice to measure one state (in this case transducer voltage) and estimate the remaining states using a standard Luenberger observer (Luenberger,
\[
\frac{d}{dt} \hat{x}(t) = A \hat{x}(t) + B i(t) - L (v(t) - B^T \hat{x}(t)) \tag{3.55}
\]

where \( \hat{x}(t) \) is the estimated value of the state vector \( x(t) \) and \( L \) is the observer gain. Designing \( L \) can be accomplished through standard pole placement methods to satisfy bandwidth constraints, or by assuming a certain measurement noise model and finding the corresponding Kalman filter gain (Kalman and Bucy, 1961). Once the \( K \) and \( L \) matrices have been determined, we can solve for the optimal dynamic admittance \( Y_c(t) \) by invoking the separation principle. In the Laplace domain, this relationship is

\[
Y_c(s) = K (sI - A - BK - LB^T)^{-1} L . \tag{3.56}
\]

It should be noted that \( Y_c(s) \) is of the same order as the augmented system.

To illustrate the design of the optimal admittance \( Y_c(s) \), we design the observer gain using standard pole placement methods such that the poles of the observer are two times the real part of the open loop poles of the augmented system. For this example, we fix \( \zeta_a = 0.5 \) and examine how the magnitude and phase of \( Y(j\omega) \) (where \( s = j\omega \)) changes for various levels of \( R \) and \( d \). An illustration of this is presented in Figure 3.5 for three levels of \( R \). It is interesting to note that the optimal dynamic admittance

![Figure 3.5: Magnitude and phase of \( Y_c(j\omega) \) for \( \zeta_a = 0.5 \) and for \( R = 0.01 \) (solid), \( R = 0.1 \) (dash), and \( R = 1 \) (dash-dot): (a) \( d = 0.01 \); (b) \( d = 0.1 \).]
admittance transfer function resembles a notch filter. Designing a notch filter using passive circuit components and op-amps would be straight-forward for a fourth order system. In an experimental realization of the optimal dynamic admittance, the notch filter would replace the block denoted by “Y” in Figure 3.3(a).

3.5 Enhanced $\bar{P}_{gen}$ Performance

The ratio of the average power generated with the electronics implementing the optimal static admittance over the average power generated with the electronics implementing the optimal dynamic admittance, gives us an idea of the potential for improvement in energy harvesting performance. We will refer to this ratio as the “$\bar{P}_{gen}$ ratio”. We now present results illustrating the $\bar{P}_{gen}$ ratio for the simple electromagnetic energy harvesting example. Figure 3.6 shows the $\bar{P}_{gen}$ ratio for various values of $d$ and $R$, and for ranges of $\zeta_a \in [0, 1]$. From these plots we see that there is a finite bandwidth for $a(t)$ at which knowledge of the derivative of the harvester states together with the disturbance acceleration is most beneficial. Furthermore, we see that knowledge of these states greatly improves performance over the entire range of $\zeta_a$ values for the asymptotic case where $R \to 0$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure36.png}
\caption{$\bar{P}_{gen}$ ratio for $R$ values of 0, 0.05, 0.08, 0.14, 0.22, 0.37, 0.61, 1 (from bottom to top): (a) $d = 0.01$; (b) $d = 0.1$.}
\end{figure}
It is interesting to note for the case where \( \zeta_a \to 0 \) that the ratio is equal to unity for every value of \( R \) and \( d \). This is due to the fact that when the disturbance is purely narrowband and tuned to the natural frequency of the harvester, the optimal way to absorb power is to have the electronics simulate a static admittance across the terminals of the transducer. In other words, the optimal way to absorb power is to have the transducer apply an electromechanical force that is proportional to the velocity of the harvester. However, if the resonance of the harvester is not tuned to be at the frequency of the disturbance, then the optimal dynamic admittance will significantly outperform the optimal static admittance. To illustrate this fact, we re-define the disturbance dynamics state space as

\[
A_a = \begin{bmatrix} 0 & 1 \\ -\omega_a^2 & -2\omega_a\zeta_a \end{bmatrix}, \quad (3.57)
\]

\[
B_a = \begin{bmatrix} 0 \\ 2\sqrt{\zeta_a\omega_a} \end{bmatrix}^T, \quad (3.58)
\]

\[
C_a = \begin{bmatrix} 0 & 1 \end{bmatrix}. \quad (3.59)
\]

where \( \omega_a \) centers the nondimensional frequency of the disturbance at a value that is no longer equal to unity. The plot in Figure 3.7 show the \( \bar{P}_{gen} \) ratio for various values of \( d \) and \( R \), and for ranges of \( \omega_a \in [0, 4] \).

Both of the plots in Figure 3.7 illustrate some interesting features of the \( \bar{P}_{gen} \) ratio for the case where \( \zeta_a \to 0 \). Firstly, we see that as \( \omega_a \to 0 \), every ratio except for the one corresponding to \( R \to 0 \) approaches unity. This can be attributed to the fact that no power is being harvested at this limit. Secondly, as \( \omega_a \to \infty \), the \( \bar{P}_{gen} \) ratios approach unity for every value of \( R \) except for \( R \to 0 \). From these plots we can conclude that the dynamic admittance is much more robust to changes in \( \omega_a \). This makes the dynamic admittance more effective than the static admittance for real-world energy harvesting applications where the disturbance is a narrowband stochastic process centered about a known frequency with some variance.
Figure 3.7: $\bar{P}_\text{gen}$ ratio for $\zeta_a \to 0$ and for $R$ values of 0, 0.05, 0.08, 0.14, 0.22, 0.37, 0.61, 1 (from bottom to top): (a) $d = 0.01$; (b) $d = 0.1$. 
Partial-State Feedback Optimization

In this chapter we begin by presenting a theorem on the decoupling properties of the Riccati equation for the optimal state feedback control of an energy harvesting system. We show that appropriate tuning of the passive parameters in the harvesting system results in a decoupled solution to the Riccati equation and a corresponding controller that only requires half of the states for feedback. For the SDOF energy harvesting example, the solution to the Riccati equation has a closed form. Thus, the corresponding expressions for the feedback control law and average power generated also have closed form solutions. From the closed form solution to the feedback controller, we present an analog circuit which could be used to implement a feedback control law by summing static gains of voltage and acceleration measurements. The realization of this circuit can be achieved using only op-amps and passive circuit components.

When tuning methods cannot be used to facilitate the decoupling property of the Riccati equation, it is possible to determine the states in the feedback law that contribute the most to the average power generated by the harvester. As such, we show that these partial-state feedback gains can be optimized using a gradient
descent method. To illustrate the optimization of the partial-state feedback gains, we derive the equations of motion for a two-degree-of-freedom tuned mass damper (TMD), which does not satisfy the decoupling properties. The benefits of using the optimized partial-state feedback controller over the optimal static admittance are presented by comparing their respective $P_{gen}$ ratios.

4.1 Partial-State Feedback Controller

Implementing the state feedback control law in Equation (3.40) generally requires knowledge or estimation of every state in the system. In other words, every component in the feedback gain matrix $K$ will in general be nonzero. However, it turns out that if the augmented system can be expressed in the realization presented in Chapter 3, then the stabilizing solution $P$ to the Riccati equation in Equation (3.42) has a special structure. Specifically, $P$ has several block entries that are equal to zero and several non-zero block entries that are repeated. As a result of this, the solution for $K$ in Equation (3.41) then has many entries which are also zero. We now present a theorem which formally introduces these concepts.

**THEOREM 2.** If the harvester dynamics can be expressed as

\[
A_h = \begin{bmatrix} 0 & I \\ -I & -D \end{bmatrix}, \tag{4.1}
\]

\[
B_h = \begin{bmatrix} 0 & B_1^T \\ 0 & B_2^T \end{bmatrix}^T, \tag{4.2}
\]

\[
G_h = \begin{bmatrix} 0 & B_1^T \\ 0 & B_2^T \end{bmatrix}^T \tag{4.3}
\]

and the disturbance dynamics can be expressed as

\[
A_a = \begin{bmatrix} 0 & I \\ -I & -Z \end{bmatrix}, \tag{4.4}
\]

\[
B_a = \begin{bmatrix} 0 & Q^T \\ 0 & I \end{bmatrix}^T, \tag{4.5}
\]

\[
C_a = \begin{bmatrix} 0 & I \end{bmatrix} \tag{4.6}
\]
then the unique, stabilizing solution to the Riccati equation in Equation (3.42) is

\[
P = \begin{bmatrix} P_{22} & 0 & P_{24} & 0 \\ 0 & P_{22} & 0 & P_{24} \\ P_{24}^T & 0 & P_{44} & 0 \\ 0 & P_{24}^T & 0 & P_{44} \end{bmatrix}
\] (4.7)

where \( P_{22}, P_{24}, \) and \( P_{44} \) can be solved sequentially as

\[
P_{22}D + D^T P_{22} + \frac{1}{R} (P_{22} + \frac{1}{2}I) B_1 B_1^T (P_{22} + \frac{1}{2}I) = 0
\] (4.8)

\[
P_{22} B_2 - P_{24} Z - D^T P_{24} - \frac{1}{R} (P_{22} + \frac{1}{2}I) B_1 B_1^T P_{24} = 0
\] (4.9)

\[
P_{24}^T B_2 + B_2^T P_{24} - P_{44} Z - Z^T P_{44} - \frac{1}{R} P_{24}^T B_1 B_1^T P_{24} = 0.
\] (4.10)

For the state vector partitioned analogously to \( P \), the corresponding optimal gain matrix is

\[
K = \begin{bmatrix} 0 & -\frac{1}{R} B_1^T (P_{22} + \frac{1}{2}I) & 0 & -\frac{1}{R} B_1^T P_{24} \end{bmatrix}
\] (4.11)

and the optimal power generation is

\[
\bar{P}_{gen} = -Q^T P_{44} Q
\] (4.12)

Proof. We begin by defining the unique, stabilizing solution to the Riccati equation in Equation (3.42) as

\[
P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}
\] (4.13)

where

\[
P_1 = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}
\] (4.14)

\[
P_2 = \begin{bmatrix} P_{13} & P_{14} \\ P_{23} & P_{24} \end{bmatrix}
\] (4.15)

\[
P_3 = \begin{bmatrix} P_{33} & P_{34} \\ P_{34}^T & P_{44} \end{bmatrix}
\] (4.16)
Thus, for the augmented state space system in Equation (3.13), the Riccati equation can be expanded into three coupled equations in $P_1$, $P_2$, and $P_3$; i.e.,

$$P_1 A_h + A_h^T P_1 - \frac{1}{R} \left( P_1 + \frac{1}{2} I \right) B_h B_h^T \left( P_1 + \frac{1}{2} I \right) = 0$$  \hspace{1cm} (4.17)$$

$$P_1 G_h C_a + P_2 A_a + A_a^T P_2 - \frac{1}{R} \left( P_1 + \frac{1}{2} I \right) B_h B_h^T P_2 = 0$$  \hspace{1cm} (4.18)$$

$$P_2^T G_h C_a + C_a^T G_h^T P_2 + P_3 A_a + A_a^T P_3 - \frac{1}{R} P_2^T B_h B_h^T P_2 = 0.$$  \hspace{1cm} (4.19)$$

As discussed in Scruggs (2010), if the harvester is WSPR, then Equation (4.17) has a unique solution $P_1$ which is stabilizing (i.e., for which $A_h - \frac{1}{R} B_h^T \left( P_1 + \frac{1}{2} I \right)$ is asymptotically stable), and for which $P_1 + \frac{1}{2} I > 0$ is a closed-loop Lyapunov matrix for the free response of the controlled harvester (i.e., with $a(t) = 0$). It only remains to show that the decoupled solution above is in fact the stabilizing one. Defining $\bar{P}_1 = P_1 + \frac{1}{2} I$, Equation (4.17) becomes

$$- \frac{1}{2} \left( A_h + A_h^T \right) + \bar{P}_1 A_h + A_h^T \bar{P}_1 - \frac{1}{R} \bar{P}_1 B_h B_h^T \bar{P}_1 = 0$$  \hspace{1cm} (4.20)$$

For the case where $A_h$ and $B_h$ satisfy Equations (4.1) and (4.2), then it is straightforward to verify that a solution to Equation (4.20) is $P_{11} = P_{22} = \bar{P}_{22}$ and $P_{12} = 0$ where $\bar{P}_{22}$ can be found by solving the Riccati equation

$$\frac{1}{2} \left( D + D^T \right) + \bar{P}_{22} \left( -D \right) + \left( -D \right)^T \bar{P}_{22} - \frac{1}{R} \bar{P}_{22} B_1 B_1^T \bar{P}_{22} = 0$$ \hspace{1cm} (4.21)$$

which is equivalent to Equation (4.8) for $P_1$. Now, it is a fact that $\left( -D, D + D^T \right)$ is observable if $(A_h, A_h + A_h^T)$ is. This follows from the fact that if $v$ is an eigenvector of $A_h$ with eigenvalue $\eta$ with $\Re\{\eta\} < 0$, then it must have the structure $v = \left[ v_1^T \eta v_1^T \right]^T$, which, in turn, implies that $v_1$ is an eigenvector of $-D$, with associated eigenvalue $\eta_1 = \eta + 1/\eta$. If $\left( -D, D + D^T \right)$ were unobservable, that would require that at least one such $v_1$ satisfy $(D + D^T)v_1 = 0$, but this would also require that the
corresponding \( v \) satisfy \((A_h + A_h^T)v = 0\), which violates the WSPR assumption. By contradiction, we conclude that \((-D, D + D^T)\) is observable. Furthermore, in the process we have shown that \(-D\) is asymptotically stable, because if \(\Re\{\eta\} < 0\) then \(\Re\{\eta_1\} = \Re\{\eta\} + \Re\{1/\eta\} < 0\). But if \(-D\) is asymptotically stable, \((-D, D + D^T)\) is observable, and \(R > 0\), then Equation (4.21) is a standard Riccati equation with a unique stabilizing solution \(P_{22} = P_{22}^T > 0\) (Bernstein, 2009). Consequently, we have that there is a corresponding unique stabilizing solution \(P_{22} = P_{22}^T\) to Equation (4.8), and that the corresponding decoupled solution for \(P_1 = P_1^T\) to Equation (4.17) is the stabilizing solution.

Now, as shown in Scruggs (2010), the other components of \(P\) are found uniquely, once \(P_1\) is found. Using the decoupled result obtained for \(P_1\) and substituting \(A_a\) and \(C_a\) as defined in Equations (4.4) and (4.6) into Equation (4.18) results in \(P_{14} = P_{23} = 0\). In addition, we have that \(P_{13} = P_{24}\) where \(P_{24}\) can be found by solving Equation (4.9). With \(P_1\) and \(P_2\) solved and the decoupled expression for \(P_2\) substituted into Equation (4.19), we have that \(P_{34} = 0\) and \(P_{33} = P_{44}\) where \(P_{44}\) is the solution to Equation (4.10). See Scruggs (2010) for the proof that \(P_3 < 0\). Substituting the decoupled solution for \(P\) into Equation (3.41) gives Equation (4.11) immediately. Finally, substituting the decoupled form for \(P\) into the expression for the upper bound on the average power generated results in Equation (4.12).

From this theorem we obtain the interesting result that only half of the states are required for the optimal energy harvesting current.

### 4.2 SDOF Example

To illustrate the decoupling properties of the Riccati equation, we return to the SDOF resonant oscillator with electromagnetic coupling in Figure 3.2. Recall that the nondimensional equations that describe the dynamics of the augmented system
are
\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & -d & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & -2\zeta_a \\
\end{bmatrix}, \quad (4.22)
\]
\[
B = [0 \; 1 \; 0 \; 0]^T, \quad (4.23)
\]
\[
G = [0 \; 0 \; 0 \; 2\sqrt{\zeta_a}]^T. \quad (4.24)
\]

From these equations it is clear that satisfy the conditions in Theorem 2. As such, the corresponding decoupled solution to the Riccati equation is
\[
P = \begin{bmatrix}
P_{22} & 0 & P_{24} & 0 \\
0 & P_{22} & 0 & P_{24} \\
P_{24} & 0 & P_{44} & 0 \\
0 & P_{24} & 0 & P_{44} \\
\end{bmatrix}. \quad (4.25)
\]

We can solve for \(P_{22}, P_{24},\) and \(P_{44}\) by sequentially solving Equations (4.8)–(4.10); i.e.,
\[
2dP_{22} + \frac{1}{R}(P_{22} + \frac{1}{2})(P_{22} + \frac{1}{2}) = 0 \quad (4.26)
\]
\[
P_{22} - P_{24}(2\zeta_a + d) - \frac{1}{R}(P_{22} + \frac{1}{2})P_{24} = 0 \quad (4.27)
\]
\[
2P_{24} - 4\zeta_a P_{44} - \frac{1}{R}P_{24}^2 = 0 \quad (4.28)
\]

In this case we can find the closed form, negative definite solution to Equation (4.25) as
\[
P_{22} = -\left(\frac{1}{2} + dR\right) + \sqrt{d^2R^2 + dR} \quad (4.29)
\]
\[
P_{24} = R\left(\frac{dR + \frac{1}{2}}{2\zeta_a R + \sqrt{d^2R^2 + dR}}\right) \quad (4.30)
\]
\[
P_{44} = \frac{1}{4\zeta_a}\left(2R - \frac{dR + \frac{1}{2}}{2\zeta_a R + \sqrt{d^2R^2 + dR}}\right) - \left(\frac{dR + \frac{1}{2}}{2\zeta_a R + \sqrt{d^2R^2 + dR}}\right)^2 \quad (4.31)
\]
Thus, we have a symbolic solution for the optimal energy harvesting current; i.e.,

\[ i(t) = \left( \frac{dR - \sqrt{d^2 R^2 + dR}}{R} \right) v(t) - \left( -\frac{(dR + \frac{1}{2}) + \sqrt{d^2 R^2 + dR}}{2R\zeta_a + \sqrt{d^2 R^2 + dR}} \right) a(t) \quad (4.32) \]

where we have used the fact that \( v(t) \) and \( a(t) \) are the second and fourth states in \( x(t) \), respectively. Furthermore, we have a symbolic solution for the corresponding optimal harvested power, as

\[ \bar{P}_{gen} = \left( -\frac{2R}{2R\zeta_a + \sqrt{d^2 R^2 + dR}} + \left( -\frac{(dR + \frac{1}{2}) + \sqrt{d^2 R^2 + dR}}{2R\zeta_a + \sqrt{d^2 R^2 + dR}} \right)^2 \right) \quad (4.33) \]

It is interesting to examine the symbolic dependency of the optimal feedback law in Equation (4.32), on the parameters \( d, \zeta_a \), and \( R \). Referring to the gains \( K_v \) and \( K_a \) in Equation (4.32) (i.e., \( i(t) = K_v v(t) + K_a a(t) \)), we notice that as \( R \to 0 \), both these gains go to infinity. It is also interesting that \( K_v \) is independent of the bandwidth of \( a(t) \). Meanwhile, \( K_a \) reduces in magnitude from its value for harmonic excitation (with \( \zeta_a = 0 \)) to the infinite-broadband case, for which \( K_a \to 0 \). For the narrowband case, \( K_a \) is significantly nonzero, implying that even when \( a(t) \) is nearly harmonic, it is still the case that explicit knowledge of \( a(t) \) may be leveraged to improve harvesting performance. However, in the limit as \( \zeta_a \to 0 \), both \( a(t) \) and \( v(t) \) become purely sinusoidal, and exactly in phase. (The phase condition is a consequence of the fact that the harvester is assumed to be tuned to the center of the passband for \( a(t) \).) Thus, in this limiting case, knowledge of both \( v(t) \) and \( a(t) \) is redundant, as one is known to be a scaled version of the other. We can therefore conclude that in this case, the optimal \( i(t) \) is in fact attained by imposing a static admittance; i.e., \( i(t) = K_v v(t) \). This observation is harmonious with what we expect from approaching the harmonic energy harvesting problem for tuned harvesters from an impedance matching perspective (Scruggs, 2010).
The optimal average power generated, as expressed in Equation (4.33), is always positive. Furthermore, we see that it increases monotonically as $\zeta_a$ decreases. This is to be expected, as it stands to reason that for disturbance models of equal acceleration intensity, ones with more signal strength concentrated near resonance will be easier to harvest energy from. Although it is less obvious from Equation (4.33), it is also the case that the power generation decreases monotonically as $R$ increases. This is also to be expected, as it stands to reason that as the electronics become less efficient, the harvesting potential decreases.

As pointed out earlier in this chapter, it is possible to tune passive components in order to satisfy the conditions in Theorem 2 for certain energy harvesting systems. For example, the optimal controller for a base excited piezoelectric bimorph energy harvester (where the dynamics of the cantilever beam are presumed to be dominated by the fundamental mode) can be made to decouple by including an inductor in parallel at the terminals of the patches. In Cassidy et al. (2011b), it is shown that tuning the ratio of the natural frequency of the beam and the natural frequency of the equivalent electrical circuit of the piezoelectric patch coupled to a parallel inductor such that they are equal will result in the Riccati equation decoupling. Furthermore, it is shown that there are certain regions in the $\{\zeta_a, R\}$ domain at which this tuned solution results in the optimal $\bar{P}_{gen}$ performance over the range of possible inductance values.

4.2.1 Partial-State Controller Circuit Realization

Although the analysis in this chapter is primarily theoretical, we pause now to consider the hypothetical implementation of feedback law in Equation (4.32). A diagram depicting one possible implementation is shown in Figure 4.1. As shown, the H-bridge is used to impose the control current for a single-phase transducer. However, it would be straightforward to extend this realization for a three-phase transducer.
such that the feedback law determines the quadrature control current, which would then be tracked by a three-phase H-bridge (or servo drive). It should be noted that the supply power is denoted by $V_S$ while the logic power is denoted by $V_L$. There are several components in this diagram. The first component is the H-bridge driver, which is used to control the four MOSFETs ($Q_1$, $Q_2$, $Q_3$, and $Q_4$) in the bridge. A typical H-bridge driver requires two inputs: a PWM signal and a triangle wave signal, which is not pictured. The driver accomplishes tracking of a desired current command signal by switching the MOSFETs on and off at a high frequency using PWM such that the average value of the current imposed across the terminals of the transducer is approximately equal to the command signal.

The next components in this circuit are the voltage, current, and acceleration sensing. The differential voltage across the terminals of the transducer is measured by a simple differential amplifier. Similarly, the current flowing into or out of the transducer can be sensed by measuring the differential voltage across a low resistance sensing resistor, $R_{\text{sense}}$. An accelerometer is attached to the base of the structure and is used to output a voltage signal that is proportional to the disturbance acceleration. Next, the voltage and acceleration signals are sent to two inverting amplifiers that multiply the signals by their respective optimal partial-state gains. These gains can

---

**Figure 4.1:** Partial-state feedback circuit diagram consisting of voltage and acceleration measurements with gain adjustment.
be tuned using two potentiometers (represented by Pot 1 and Pot 2 in the figure).

The final component in this circuit is a summing amplifier with proportional and integral control in feedback. The voltage signal, acceleration signal, and current signal are summed together to produce the error signal. The error signal then passes through PI controller, which outputs the PWM signal that is sent to the H-bridge driver. If the chosen resistor values are made such that $R_1 = R_2 = R_3$ then the proportional gain is just $-R_4/R_1$. The integral gain can be adjusted by changing the capacitor value $C_1$ that is in series with $R_4$. The values of the proportional and integral gains should be tailored for the particular components used in the circuit as well as the bandwidth constraints on the current command signal.

4.3 Partial-State Feedback Gain Optimization

We now present a procedure for optimizing the feedback gains corresponding to states that are the most important in terms of the average power generated by the harvester. Since most energy harvesting systems don’t satisfy the conditions in Theorem 2, they require knowledge of all of the states for optimal power generation. For practical purposes, it is often the case that measuring all of the states is not feasible. Instead, transducer voltage measurements can be passed through a Luenberger observer (Luenberger, 1971), which is used to estimate the remaining system states. Observer gains can be chosen to achieve close tracking of states, but the construction of a dynamic observer also complicates the feedback circuitry.

It is possible to achieve performance that is almost as good as the optimal full state upper bound by directly measuring and feeding back only the states that have the most influence on the performance. We have seen in the previous section that when the Riccati equation decouples, only the derivatives of the harvester electromechanical coordinates, and the disturbance acceleration (but not its integral) are required for feedback. From this insight we presume that these states have the most
influence on performance even when the Riccati equation does not decouple. Furthermore, these states may sometimes be the easiest to measure. Implementing circuitry to sum static gains of measured states is very simple.

In order to solve for the partial-state feedback gains, we note that the output of Equation (3.13) is no longer just voltage. Instead, we define a new output vector \( y(t) = Cx(t) \). The matrix \( C \) is defined such that \( y(t) \) contains the states that we choose to include for feedback. In other words, \( C \) should be normalized such that \( CC^T = I \). As such, we define the partial-state current input relationship as

\[
i(t) = \tilde{K}Cx(t)
\]

where \( \tilde{K} \) is a vector of the optimal partial-state feedback gains. Next, we substitute the relationship for \( i(t) \) into the performance measure \( J \); i.e.,

\[
J = \mathcal{E}\left\{x^T(t) \left[ \begin{array}{c} I \\ \frac{1}{2}C^T \tilde{K}^T \\ \frac{1}{2}B^T \frac{1}{2}B + R \\ \frac{1}{2}B \tilde{K} C \end{array} \right] x(t) \right]\}

(4.35)

\[
= \mathcal{E}\left\{x^T(t)\tilde{Q}(\tilde{K}) x(t) \right\}
\]

(4.36)

where

\[
\tilde{Q}(\tilde{K}) = \frac{1}{2}C^T \tilde{K}^T B^T + \frac{1}{2}B \tilde{K} C + R C^T \tilde{K}^T \tilde{K} C .
\]

(4.37)

In this case we have that the optimal upper bound on power generation is

\[
\tilde{P}_{gen} = -G^T \hat{P} G
\]

(4.38)

where \( \hat{P} \) is the solution to the Lyapunov equation

\[
\begin{bmatrix}
A + B \tilde{K} C \\
A + B \tilde{K} C
\end{bmatrix}^T \hat{P} + \hat{P} \begin{bmatrix}
A + B \tilde{K} C \\
A + B \tilde{K} C
\end{bmatrix} + \tilde{Q}(\tilde{K}) = 0 .
\]

(4.39)

To optimize \( \tilde{K} \), we use a gradient-descent method. It should be noted that \( \tilde{P}_{gen} \) in Equation (4.38) is nonconvex in \( \tilde{K} \), and in general there may be multiple local
minima. One of the challenges associated with this problem is the choice of the initial condition for the algorithm. A study by Cai and Lim (2005) suggest that the initial guess should be

\[ \tilde{K}_0 = KC^T \]  

(4.40)

where \( K \) is the solution to the full state Riccati equation in Equation (3.42). For the examples presented in this paper, this particular initial guess always resulted in stable closed loop dynamics.

**THEOREM 3.** The performance \( J \) in Equation (4.35) is minimized by

\[
\frac{\partial J}{\partial \tilde{K}} = 2 \left( B^T \tilde{P} + \frac{1}{2} B^T + R \tilde{K} C \right) \Sigma C^T = 0
\]

(4.41)

where \( \tilde{P} \) is the solution to the Lyapunov equation in Equation (4.39) and where \( \Sigma \) is the solution to another Lyapunov equation; i.e.,

\[
\begin{bmatrix}
A + B \tilde{K} C \\
\end{bmatrix} \Sigma + \Sigma \begin{bmatrix}
A + B \tilde{K} C \\
\end{bmatrix}^T + GG^T = 0
\]

(4.42)

**Proof.** This is a standard result from linear-quadratic control theory (Dorato et al., 1995). \( \Box \)

This theorem leads to a simple first-order gradient method for optimizing \( \tilde{K} \), consisting of the following steps for iteration \( i \).

1. Start with \( \tilde{K}_0 = KC^T \) as the initial guess.

2. For a matrix \( \tilde{K}_i \), evaluate \( \partial J / \partial \tilde{K} \) as in Equation (4.41).

3. Compute \( \tilde{K}_{i+1} \) by updating \( \tilde{K}_i \) in the direction of the steepest-descent, with a user-specified step size \( \epsilon \), as

\[
\tilde{K}_{i+1} = \tilde{K}_i - \epsilon \left. \frac{\partial J}{\partial \tilde{K}} \right|_{\tilde{K}_i}
\]

(4.43)
4. Return to Step 2 with $i \leftarrow i + 1$ until convergence is reached.

Since the focus of this study is restricted to single-transducer energy harvesting systems, the efficiency of the optimization algorithm is of little concern. The gradient descent algorithm converges to a local solution in $J$ in 10–20 iterations for the example discussed later in this chapter. This is likely due to the fact that the initial guess $\tilde{K}_0$ is close to a local minima. A more robust method for systems with an arbitrary number of transducers is presented in Iwasaki et al. (1994). The algorithm used in that study involves solving linear matrix inequalities (LMIs) within a scaled min/max optimization routine.

4.4 Tuned Mass Damper Example

Consider the tuned mass damper (TMD) energy harvester in Figure 4.2. The system consists of a SDOF resonant oscillator with an electromagnetic transducer which is coupled to the top story of a much larger SDOF resonant oscillator. The equations of motion that describe this system are

$$m_1 \ddot{r}_1(t) + (b_1 + b_2)\dot{r}_1(t) - b_2\dot{r}_2(t) + (k_1 + k_2)r_1(t) - k_2r_2(t) = m_1a(t) \quad (4.44a)$$

$$m_2 \ddot{r}_2(t) - b_2\dot{r}_1 + b_2\dot{r}_2 - k_2r_2(t) + k_2r_1(t) = m_2a(t) + c_e i(t) \quad (4.44b)$$

![Figure 4.2: Tuned mass damper energy harvester.](image-url)
where \( r_1(t) \) is the relative displacement of the structure and \( r_2(t) \) is the relative displacement of the TMD. It should be noted that the transducer voltage is just \( v(t) = c_e r_2(t) \). For simplicity we will make the assumption that the mass of the TMD is some percentage of the mass of the structure; i.e., \( m_2 = \alpha m_1 \) where \( \alpha \in (0,1) \). With this assumption, we have that \( k_2 = \alpha k_1 \) so that the natural frequency of the TMD is equal to the natural frequency of the structure. As such, Equation (4.44) can be re-expressed as

\[
m\ddot{r}_1(t) + (b_1 + b_2)\dot{r}_1(t) - b_2\dot{r}_2(t) + (1 + \alpha)k_1 t - \alpha k_2 t = ma(t) \quad (4.45a)
\]

\[
am\ddot{r}_2(t) - b_2\dot{r}_1(t) + b_2\dot{r}_2(t) - \alpha k_2 t + \alpha k_1 t = ama(t) + c_e i(t) \quad (4.45b)
\]

where we have set \( m_1 = m \) and \( k_1 = k \).

Similar to the SDOF example in the previous chapter, we make the following substitutions for time and relative displacement of the two degrees of freedom; i.e.,

\[
t = \sqrt{\frac{m}{k}} \tau, \quad (4.46)
\]

\[
r_1(t) = \left(\frac{m\sigma_a}{k}\right) \bar{r}_1(\tau), \quad (4.47)
\]

\[
r_2(t) = \left(\frac{m\sigma_a}{k}\right) \bar{r}_2(\tau). \quad (4.48)
\]

Thus, we have that the harvester dynamics can be represented in the following nondimensional coordinates

\[
\ddot{\bar{r}}_1(\tau) + (d_1 + d_2)\dot{\bar{r}}_1(\tau) - d_2\dot{\bar{r}}_2(\tau) + (1 + \alpha)\bar{r}_1(\tau) - \alpha \bar{r}_2(\tau) = \bar{a}(\tau) \quad (4.49a)
\]

\[
\alpha \ddot{\bar{r}}_2(\tau) - d_2\dot{\bar{r}}_1(\tau) + d_2\dot{\bar{r}}_2(\tau) + \alpha \bar{r}_1(\tau) - \alpha \bar{r}_2(\tau) = \alpha \bar{a}(\tau) + \bar{i}(\tau) \quad (4.49b)
\]

where the nondimensional damping factors are

\[
d_1 = \frac{b_1}{\sqrt{mk}}, \quad (4.50)
\]

\[
d_2 = \frac{b_2}{\sqrt{mk}}. \quad (4.51)
\]
The nondimensional acceleration, current, and voltage remain unchanged from the SDOF example (see Equations (3.24)–(3.26)).

Next, if we define the vector \( \mathbf{q}(\tau) = [\bar{r}_1(\tau) \; \bar{r}_2(\tau)]^T \), then Equation (4.49) can be represented in matrix form as

\[
M\ddot{\mathbf{q}}(\tau) + D\dot{\mathbf{q}}(\tau) + S\mathbf{q}(\tau) = M\Gamma_1\bar{a}(\tau) + \Gamma_2\bar{i}(\tau)
\]

(4.52)

where

\[
M = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix},
\]

(4.53)

\[
D = \begin{bmatrix} d_1 + d_2 & -d_2 \\ -d_2 & d_2 \end{bmatrix},
\]

(4.54)

\[
S = \begin{bmatrix} 1 + \alpha & -\alpha \\ -\alpha & \alpha \end{bmatrix},
\]

(4.55)

\[
\Gamma_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T,
\]

(4.56)

\[
\Gamma_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T.
\]

(4.57)

If we consider the harvester to be characterized by the state vector

\[
\mathbf{x}_h(\tau) = \begin{bmatrix} \mathbf{S}^{1/2} & 0 \\ 0 & \mathbf{M}^{1/2} \end{bmatrix} \begin{bmatrix} \mathbf{q}(\tau) \\ \dot{\mathbf{q}}(\tau) \end{bmatrix}
\]

(4.58)

then the harvester dynamics matrix and input matrices are defined as

\[
\mathbf{A}_h = \begin{bmatrix} 0 & \mathbf{S}^{1/2}\mathbf{M}^{-1/2} \\ -\mathbf{M}^{-1/2}\mathbf{S}^{1/2} & -\mathbf{M}^{-1/2}\mathbf{D}\mathbf{M}^{-1/2} \end{bmatrix},
\]

(4.59)

\[
\mathbf{B}_h = \begin{bmatrix} 0 \\ \mathbf{M}^{-1/2}\Gamma_2 \end{bmatrix},
\]

(4.60)

\[
\mathbf{G}_h = \begin{bmatrix} 0 \\ \mathbf{M}^{1/2}\Gamma_1 \end{bmatrix}.
\]

(4.61)

For this example, we have that the disturbance dynamics are nondimensionalized
by the following state space matrices

\[ \mathbf{A}_a = \begin{bmatrix} 0 & 1 \\ -\omega_a^2 & -2\omega_a \zeta_a \end{bmatrix}, \]  
\[ \mathbf{B}_a = \begin{bmatrix} 0 \\ 2\sqrt{\zeta_a \omega_a} \end{bmatrix}^T, \]  
\[ \mathbf{C}_a = \begin{bmatrix} 0 & 1 \end{bmatrix}. \]  

(4.62)  
(4.63)  
(4.64)

We note that the nondimensional frequency of the disturbance passband is centered at \( \omega_a \). The two natural frequencies of the combined structure can be found by solving for the eigenvalues of \( \mathbf{M}^{-1}\mathbf{S} \); i.e.,

\[ \lambda_{1,2} = \frac{1}{2} \left( 2 + \alpha \mp \sqrt{4\alpha + \alpha^2} \right). \]  

(4.65)

For this example, we will set \( \omega_a = \sqrt{\lambda_1} \), which corresponds to the first natural frequency of the combined structure. Finally, combining the harvester and disturbance dynamics, we can construct the augmented state space as in Equation (3.13).

Upon inspection of augmented dynamics matrix \( \mathbf{A} \), it is clear that the block matrices in (1,2) and (2,1) violate the decoupling conditions in Theorem 2. This is due to the fact that the combined system has two distinct natural frequency (i.e., \( \lambda_1 \neq \lambda_2 \)). As such, we can follow the partial-state gain optimization algorithm described in the previous section to optimize certain static gains. We assume that the disturbance acceleration and voltage produced by the transducer are the most important states in terms of the average power generation. With this assumption, we define the output matrix as

\[ \mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \]  

(4.66)

and implement the gradient descent method to determine \( \mathbf{K} \).

Before we illustrate the advantage of using the optimized partial-state feedback gains over the static admittance, we pause to define some notation. For clarity, let
\( P^f_{\text{gen}} \), \( P^y_{\text{gen}} \), and \( P^v_{\text{gen}} \) be the optimal power generation values under full-state, static partial-state, and static velocity feedback, respectively. We will refer to various ratios of these three quantities, for example, as \( \rho^{v/f} = \frac{P^v_{\text{gen}}}{P^f_{\text{gen}}} \). Analogous conventions hold for other ratios.

In Figure 4.3 we compare the \( \rho^{v/f} \) and \( \rho^{y/f} \) over the range of \( \zeta_a \in [0, 1] \) for three levels of TMD mass. Figures 4.3(a), 4.3(c), and 4.3(e) are plots of \( \rho^{v/f} \) vs. \( \zeta_a \) for \( \alpha \) values equal to 0.05, 0.1, and 0.2, respectively. Similarly, Figures 4.3(b), 4.3(d), and 4.3(f) are plots of \( \rho^{y/f} \) vs. \( \zeta_a \) for \( \alpha \) values equal to 0.05, 0.1, and 0.2, respectively. It should be noted that \( d_1 = d_2 = 0.01 \) for all six plots. From these plots, we see that including the disturbance acceleration in addition to the transducer voltage and optimizing their respective feedback gains outperforms the optimal static admittance in terms of the \( P_{\text{gen}} \) ratios. Again, we see that there is a finite bandwidth at which including the optimized acceleration gain is most beneficial. It is interesting to note that as \( \alpha \) increases, the performance of the optimal static admittance decreases with respect to the performance of the optimal full-state feedback controller. However, very little performance of the partial-state feedback controller is sacrificed as \( \alpha \) increases (with the exception of the asymptotic case where \( R \to 0 \)). In fact, the performance of the partial-state feedback controller with respect to the performance of the optimal full-state feedback controller at higher values of \( R \) seems to improve as \( \alpha \) increases.
Figure 4.3: Comparison of the $\bar{P}_{gen}$ ratios for the TMD, for $d_1 = d_2 = 0.01$ and for $R$ values of 0, 0.05, 0.08, 0.14, 0.22, 0.37, 0.61, 1 (from bottom to top): (a) $\rho^{vf}$ vs. $\zeta_a$ for $\alpha = 0.05$; (b) $\rho^{vf}$ vs. $\zeta_a$ for $\alpha = 0.05$; (c) $\rho^{vf}$ vs. $\zeta_a$ for $\alpha = 0.1$; (d) $\rho^{vf}$ vs. $\zeta_a$ for $\alpha = 0.1$; (e) $\rho^{vf}$ vs. $\zeta_a$ for $\alpha = 0.2$; (f) $\rho^{vf}$ vs. $\zeta_a$ for $\alpha = 0.2$. 
Energy Harvesting with a Nonlinear Transducer

In this chapter we combine the modeling and device characterization of the electromagnetic transducer in Chapter 2 with the optimal energy harvesting controller presented in Chapter 3. We begin by deriving the optimal resistive load for a SDOF energy harvester that is excited by a sinusoidal disturbance at single frequency using impedance matching theory. For this example, energy conversion is accomplished with the electromagnetic transducer and coupled to the servo drive from Chapter 2. The main contribution resulting from this example is that we include the nonlinear dissipative effects of the harvesting transducer in the calculations of the average power generated and the optimal resistive load. The theoretical impedance matched resistive load is validated experimentally using real-time hybrid-testing (RTHT) for three levels of passive viscous damping in the system. Both the theoretical and experimental investigations are compared and results are shown to match closely.

In the final section of this chapter, we derive the statistically linearized energy harvesting controller for a SDOF energy harvester with nonlinear Coulomb friction introduced by the harvesting transducer. We exploit the stationarity properties of the nonlinear system in order to determine an equation for the approximate stationary
covariance matrix of an equivalent linear system. It is shown that the covariance matrix can be determined by solving a nonlinear algebraic equation. Next, we use the necessary conditions from optimal control theory to derive an equation that must be satisfied to determine the optimal controller. Because these two equations are coupled, nonlinear algebraic equations, solving for the covariance matrix and optimal controller requires the implementation of an iterative algorithm. Finally, we compare the average power generated by the statistically linearized SDOF energy harvester to the average power generated by a frictionless SDOF energy harvester for both the optimal full-state controller and the optimal static admittance.

5.1 Impedance Matching of a SDOF Energy Harvester

We now present an example in which the electromagnetic transducer and servo drive are attached to a SDOF oscillator. One of the simplest ways to optimize power from such an application is by resistive impedance matching. We assume that the SDOF is excited at its base by a sinusoidal acceleration with amplitude $A_0$ and frequency $w_0$. First, we develop an analytical expression for the average power generated and use this expression to determine the optimal load resistance that should be imposed by the servo drive to maximize power. Next, we compare the analytical model to our experimental system. The electromagnetic transducer is back-driven by a hydraulic actuator, which simulates the dynamics of the SDOF oscillator using

![Figure 5.1: Hybrid testing block diagram.](image-url)
real-time hybrid-testing (RTHT). RTHT is accomplished in dSpace by feeding back force measurements from the electromagnetic transducer in real-time to a simulated disturbance and structure model. The force measurement and simulated disturbance force are summed together and used to excite the simulated structure. The output of the simulated structure is a displacement command signal, which is tracked by the digital controller that is used to operate the hydraulic actuator. Our experimental RTHT design follows the procedure in the study by Darby et al. (1999) and a block diagram describing this procedure can be seen in Figure 5.1.

The dynamics of the SDOF oscillator and coupled electromagnetic transducer can be described by the following second order differential equation

\[
\ddot{m}\dot{x}(t) + \ddot{c}\dot{x}(t) + \ddot{k}x(t) + F_f \text{sgn} (\dot{x}(t)) = mA_0 \sin(\omega_0 t) \tag{5.1}
\]

where

\[
\ddot{m} = m + m_d, \tag{5.2}
\]
\[
\ddot{c} = c + c_d + c_e, \tag{5.3}
\]
\[
\ddot{k} = k + k_d, \tag{5.4}
\]
\[
F_f = f_c + \gamma_1 + \gamma_2. \tag{5.5}
\]

It should be noted that the approximation of the total Coulomb friction \( F_f \) in Equation (5.5) takes the amplitudes of the bearing friction into account by summing them together with the ballscrew Coulomb friction. This estimate is conservative and it will have an effect on the amount of average power generated as predicted by this model. Next, we derive an expression for the equivalent viscous damping from the Coulomb friction. This can be accomplished through an energy balance approach where we set the amount of energy dissipated during one oscillation from the Coulomb friction equal to the amount of energy dissipated by an equivalent viscous
damper. This quantity can be expressed as

\[ c_{eq} = \frac{4F_f}{\pi \dot{X}} \]  \hspace{1cm} (5.6)

where \( \dot{X} \) is the velocity amplitude.

The next step is to derive an expression for the velocity amplitude of the system in Equation (5.1) through harmonic balance. This expression can be written as

\[ \dot{X} = \frac{mA_0 \omega_0}{\sqrt{(-\tilde{m}\omega_0^2 + \tilde{k})^2 + (\tilde{c} + c_{eq})^2 \omega_0^2}}. \]  \hspace{1cm} (5.7)

Since \( c_{eq} \) depends on \( \dot{X} \), the expression in Equation (5.7) is quadratic in \( \dot{X} \). Substituting the expression for \( c_{eq} \) in Equation (5.6) into Equation (5.7) and solving for \( \dot{X} \) results in

\[ \dot{X} = \frac{-4\tilde{c}F_f \omega_0^2 + \sqrt{\omega_0^2 \left(-16F_F^2 \left(-\tilde{m}\omega_0^2 + \tilde{k}\right)^2 + A_0^2 m^2 \pi^2 \left(\tilde{c}^2 \omega_0^2 + (-\tilde{m}\omega_0^2 + \tilde{k})^2\right)\right)}}{\pi \left(\tilde{c}^2 \omega_0^2 + (-\tilde{m}\omega_0^2 + \tilde{k})^2\right)} \]  \hspace{1cm} (5.8)

where we have ignored the negative root. The SDOF oscillator is tuned to resonate at the excitation frequency \( \omega_0 = \sqrt{\tilde{k}/\tilde{m}} \), so the expression for the velocity amplitude simplifies to

\[ \dot{X} = \frac{mA_0 - 4F_f/\pi}{\tilde{c}}. \]  \hspace{1cm} (5.9)

We can now express the average total power generated during one oscillation as a function of the back-emf of the motor; i.e.,

\[ \bar{P}_T = \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} \frac{e_q^2}{R_c + R_L} dt \]  \hspace{1cm} (5.10)
where \( e_q = \frac{3K_e}{2l} \dot{x}(t) \) is the quadrature back-emf of the motor. Therefore, solving for \( \bar{P}_T \) results in

\[
\bar{P}_T = \frac{9K_e^2 \dot{X}^2}{8l^2(R_c + R_L)}.
\]

(5.11)

Finally, we have that the average power generated across the load resistance \( R_L \) is

\[
\bar{P}_L = \bar{P}_T \frac{R_L}{R_c + R_L} = \frac{9K_e^2 \dot{X}^2 R_L}{8l^2(R_c + R_L)^2}.
\]

(5.12)

To determine the resistance that maximizes average power, we take the partial derivative of \( \bar{P}_L \) with respect to \( R_L \) and set this expression equal to zero. After some manipulation, the optimal resistive load can be written as

\[
R_L^* = R_c + \frac{9K_e^2}{4l^2(c + c_d)}.
\]

(5.13)

Substituting the expression for \( R_L^* \) into Equation (5.12) results in an analytical expression for the optimal power generated; i.e.,

\[
\bar{P}_L^* = \frac{9K_e^2 (\pi mA_0 - 4F_f)^2}{8\pi^2(c + c_d) (9K_e^2 + 4R_c l^2(c + c_d))}.
\]

(5.14)

It is interesting to note that the amount of Coulomb friction in the device does not influence the optimal resistive load. However, neglecting the Coulomb friction in the device will significantly overestimate the amount of power that can be harvested given any resistive load. In other words, including Coulomb friction effects is crucial to accurately predicting the amount of power that can be harvested from a SDOF oscillator by an electromagnetic transducer.

As previously mentioned, we compare the analytical expression for the average power generated across the load resistance \( R_L \) with the experimental system using RTHT. Three levels of passive viscous damping \( \zeta_0 \) are used in the simulated SDOF
oscillator where we define $\zeta_0$ as

$$\zeta_0 = \frac{c + c_d}{2\sqrt{\tilde{m}\tilde{k}}}.$$

The three levels of damping that we have chosen to investigate are $\zeta_0 = 0.15$, $\zeta_0 = 0.2$, and $\zeta_0 = 0.3$. Although these levels of damping are higher than typical levels for a TMD in a tall building, they illustrate three $\bar{P}_L$ vs. $R_L$ curves with three distinct optimal values. We note that the acceleration amplitude $A_0$ was set at 18cm/s$^2$ with a frequency of 0.5Hz so that the response of the SDOF oscillator would be well within the displacement and velocity limits of the hydraulic actuator. For each test at a given $R_L$ value, we collect data for 20 seconds at a sample rate of 1000Hz.

The analytical expression for the predicted average power generated across $R_L$ is determined by using optimal parameter values for the proposed model in Table 2.3. We compare the predicted power to the experimental RTHT results in Figure 5.2 for three levels of mechanical damping in the SDOF oscillator. We can see from this plot that we have excellent agreement between the predicted and experimental average power values over the range of load resistances and for the three levels of mechanical damping. The optimal load resistances given in Equation (5.13) were calculated to be 38Ω, 55Ω, and 73Ω while the optimal power generated values given in Equation (5.14) were calculated to be 2.62W, 4.02W, and 5.42W for $\zeta_0 = 0.3$, $\zeta_0 = 0.2$, and $\zeta_0 = 0.15$, respectively. In addition, we plot how the optimal load resistance $R^*_L$ and the optimal average power generated $\bar{P}^*_L$ varies for a range of $\zeta_0 \in [0.05, 0.5]$ in Figure 5.3. As expected, both values decrease monotonically as $\zeta_0$ increases.
Figure 5.2: Comparison of predicted and experimental average power generated $\bar{P}_L$ over a range of load resistance $R_L$ values: (a) $\zeta_0 = 0.15$; (b) $\zeta_0 = 0.2$; (c) $\zeta_0 = 0.3$.

Figure 5.3: Plot of the optimal load resistance $R_L^*$ and the corresponding optimal average power generated $\bar{P}_L^*$ over a range of $\zeta_0$ values: (a) $R_L^*$ vs. $\zeta_0$; (b) $\bar{P}_L^*$ vs. $\zeta_0$. 

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5.2 Statistically Linearized Energy Harvesting

5.2.1 Stationary Covariance

The general state space model for an energy harvesting system with nonlinearities introduced by the transducer is

\[
\frac{d}{dt} x(t) = Ax(t) + f(x(t), t) + Bi(t) + Gw(t) \\
v(t) = B^T x(t) \\
y(t) = Cx(t)
\] (5.16a, 5.16b, 5.16c)

where we assume the function \( f(x(t), t) \) is nonlinear. We assume \( f(0, t) = 0 \), and that it is anti-symmetric; i.e., \( f(-x(t), t) = -f(x(t), t) \). In addition, we assume that \( x(t) \) has a stationary probability distribution which can be approximated as Gaussian with zero mean (because \( f(x(t), t) \) is assumed to be anti-symmetric) and covariance \( \Sigma = \mathcal{E}\{x(t)x^T(t)\} \). The corresponding stationary probability density function (pdf) is

\[
p(x(t), t) = \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} \exp\left\{-\frac{1}{2}x^T(t)\Sigma^{-1}x(t)\right\}. \quad (5.17)
\]

If we implement the full-state feedback control law

\[
i(t) = Kx(t)
\] (5.18)

then the stationary solution to the stationary covariance \( \Sigma \) can be found via statistical linearization (Roberts and Spanos, 2003). As such, the equation that \( \Sigma \) must satisfy in stationarity is

\[
\mathcal{E}\{\nabla_x^T f_{cl}^T(x(t), t)\}^T \Sigma + \Sigma \mathcal{E}\{\nabla_x^T f_{cl}^T(x(t), t)\} + GG^T = 0 \quad (5.19)
\]

where the expectation of the function \( \psi(x(t), t) \) is

\[
\mathcal{E}\{\psi(x(t), t)\} = \int_x p(x(t), t)\psi(x(t), t) \, dx \quad (5.20)
\]
and where
\[ f_{cl}(\mathbf{x}(t),t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{k}(t) + f(\mathbf{x}(t),t) \]  

(5.21)

It should be noted that \( \nabla_x \) is the gradient operator with respect to the variable \( \mathbf{x} \).

For the case where we include the Coulomb friction present in the electromagnetic transducer, we have the closed-loop nonlinear force is
\[ f_{cl}(\mathbf{x}(t),t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{k}(t) + \mathbf{F}\text{sgn}\{y(t)\} \]  

(5.22)

where \( \mathbf{F} \) is a vector corresponding to how the Coulomb friction is introduced into the system and \( y(t) \) can be related to the relative velocity of the transducer by \( \mathbf{C} \).

If we take the gradient of \( f_{cl}(\mathbf{x}(t),t) \) with respect to \( \mathbf{x}(t) \), then we have that
\[ \nabla_x f_{cl}^T(\mathbf{x}(t),t) = \mathbf{A}^T + \mathbf{K}^T\mathbf{B}^T + \left( \frac{\partial}{\partial y}\text{sgn}\{y(t)\} \right) (\nabla_x^T y(t)) \mathbf{F}^T \]  

(5.23)

\[ = \mathbf{A}^T + \mathbf{K}^T\mathbf{B}^T + 2\delta(\mathbf{C}\mathbf{x}(t))\mathbf{C}^T\mathbf{F}^T \]  

(5.24)

where \( \delta \) is the Dirac delta function. Next, taking the expectation of both sides of Equation (5.24) results in the following simplifications
\[ \mathcal{E}\{\nabla_x f_{cl}^T(\mathbf{x}(t),t)\} = \mathbf{A}^T + \mathbf{K}^T\mathbf{B}^T + 2\mathbf{C}^T\mathbf{F}^T \int_{\mathbf{x}} \delta(\mathbf{C}\mathbf{x}(t))p(\mathbf{x}(t),t) \, d\mathbf{x} \]  

(5.25)

\[ = \mathbf{A}^T + \mathbf{K}^T\mathbf{B}^T + 2\mathbf{C}^T\mathbf{F}^T \int_{\mathbf{y}} \delta(y(t))p(y(t),t) \, dy \]  

(5.26)

\[ = \mathbf{A}^T + \mathbf{K}^T\mathbf{B}^T + 2\mathbf{C}^T\mathbf{F}^T p(y(t),t)|_{y(t)=0} . \]  

(5.27)

The pdf for \( y(t) \) is assumed to be a zero-mean Gaussian function with scalar variance \( \sigma_y^2 = \mathbf{C}\Sigma\mathbf{C}^T \); i.e.,
\[ p(y(t),t) = \frac{1}{\sqrt{2\pi\sigma_y}} \exp\left\{-\frac{1}{2}y^2(t)/\sigma_y^2\right\} . \]  

(5.28)

Thus, we have that Equation (5.27) is
\[ \mathcal{E}\{\nabla_x f_{cl}^T(\mathbf{x}(t),t)\} = \mathbf{A}^T + \mathbf{K}^T\mathbf{B}^T + \frac{2}{\sqrt{2\pi\mathbf{C}\Sigma\mathbf{C}^T}}\mathbf{C}^T\mathbf{F}^T . \]  

(5.29)
Substituting Equation (5.29) into Equation (5.19) results in the following nonlinear algebraic equation

\[
\begin{bmatrix} A + BK \end{bmatrix} \Sigma + \Sigma \begin{bmatrix} A + BK \end{bmatrix}^T + \sqrt{\frac{2}{\pi}} \frac{FC \Sigma + \Sigma C^T F^T}{\sqrt{C \Sigma C^T}} + GG^T = 0.
\]

(5.30)

5.2.2 Stationary Energy Harvesting

Recall that the energy harvesting objective is to minimize the negative of the average power generated by the harvester; i.e.,

\[
\bar{P}_{gen} = -\mathcal{E} \{ i(t) v(t) + R i^2(t) \} = -\mathcal{E} \{ \frac{1}{2} B^T x(t) i(t) + \frac{1}{2} x^T(t) B i(t) + R i^2(t) \} = -\text{tr} \left\{ \left[\frac{1}{2} K^T B + \frac{1}{2} BK + R K^T K \right] \Sigma \right\}
\]

(5.31)

over all \( K \) subject to the constraint in Equation (5.30). Next, we define the Hamiltonian \( \mathcal{H}(K, \Sigma, P) \) as

\[
\mathcal{H}(K, \Sigma, P) = \text{tr} \left\{ \left[\frac{1}{2} K^T B + \frac{1}{2} BK + R K^T K \right] \Sigma + P \left( [A + BK] \Sigma + \Sigma [A + BK]^T + \sqrt{\frac{2}{\pi}} \frac{FC \Sigma + \Sigma C^T F^T}{\sqrt{C \Sigma C^T}} + GG^T \right) \right\}
\]

(5.32)

where \( P = P^T \) is a matrix of Lagrange multipliers. Thus, we the following minimax problem

\[
\min_K \min_{\Sigma} \max_P \mathcal{H}(K, \Sigma, P).
\]

(5.33)

To find the optimal solution to the problem in Equation (5.35), we take the partial derivative of the Hamiltonian with respect to each of the decision variables and set these quantities equal to zero. This procedure constitutes a standard approach to solving an optimal control problem (Lewis and Syrmos, 1995). We start by taking
the partial derivative of $\mathcal{H}$ with respect to $\Sigma$; i.e.,

$$\frac{\partial \mathcal{H}}{\partial \Sigma_{lm}} = \text{tr} \left\{ \left[ \frac{1}{2} K^T B^T + \frac{1}{2} BK + RK^T K \right] \hat{e}_l \hat{e}_m^T + P \left( [A + BK] \hat{e}_l \hat{e}_m^T + \hat{e}_l \hat{e}_m^T [A + BK]^T \right) 
+ \sqrt{\frac{2}{\pi}} \frac{FC \hat{e}_l \hat{e}_m^T + \hat{e}_l \hat{e}_m^T C^T F^T}{\sqrt{C\Sigma C^T}} 
- \sqrt{\frac{1}{2\pi}} \frac{FC \Sigma + \Sigma C^T F^T}{(C\Sigma C^T)^{3/2}} \hat{e}_l \hat{e}_m^T C^T \right\} \hat{e}_l \hat{e}_m^T.$$  

(5.36)

where $\hat{e}_l$ and $\hat{e}_m$ are unit vectors in the $l$ and $m$ directions, respectively. Recognizing the property that $\text{tr}(AB) = \text{tr}(BA)$, we can express Equation (5.36) as

$$\frac{\partial \mathcal{H}}{\partial \Sigma_{lm}} = \hat{e}_l \hat{e}_m^T \left\{ \frac{1}{2} K^T B^T + \frac{1}{2} BK + RK^T K \right\} \hat{e}_l \hat{e}_m^T + P \left( [A + BK] \hat{e}_l \hat{e}_m^T + \hat{e}_l \hat{e}_m^T [A + BK]^T \right) 
+ \sqrt{\frac{2}{\pi}} \frac{FC \hat{e}_l \hat{e}_m^T + \hat{e}_l \hat{e}_m^T C^T F^T}{\sqrt{C\Sigma C^T}} \hat{e}_l \hat{e}_m^T C^T \hat{e}_l \hat{e}_m^T.$$  

(5.37)

If we define

$$U(\Sigma) = \frac{1}{2} C^T C \Sigma \frac{1}{C\Sigma C^T}$$  

(5.38)

$$V(\Sigma) = \sqrt{\frac{2}{\pi}} \frac{FC}{\sqrt{C\Sigma C^T}}$$  

(5.39)

$$T(K, \Sigma) = A + BK + V$$  

(5.40)

then Equation (5.37) simplifies to

$$\frac{\partial \mathcal{H}}{\partial \Sigma_{lm}} = \hat{e}_l \hat{e}_m^T \left\{ \frac{1}{2} K^T B^T + \frac{1}{2} BK + RK^T K + PT + T^T P - UPV - V^T PU^T \right\} \hat{e}_l \hat{e}_m^T.$$  

(5.41)

Thus, enforcing the condition that Equation (5.41) is equal to zero for all $\{l,m\}$, gives an equation for $P$ in terms of $\Sigma$ and $K$ as

$$\frac{1}{2} K^T B^T + \frac{1}{2} BK + RK^T K + PT + T^T P - UPV - V^T PU^T = 0.$$  

(5.42)

Next, we take the partial derivative of $\mathcal{H}$ with respect to $K$; i.e.,

$$\frac{\partial \mathcal{H}}{\partial K_{lm}} = \text{tr} \left\{ \left[ \frac{1}{2} \hat{e}_m \hat{e}_l^T B^T + \frac{1}{2} B \hat{e}_l \hat{e}_m^T + 2RK^T \hat{e}_l \hat{e}_m^T \right] \Sigma + 2P B \hat{e}_l \hat{e}_m^T \right\}.$$

(5.43)
If we circulate the arguments in the trace in Equation (5.43), then we have

$$\frac{\partial H}{\partial K_{lm}} = \hat{e}_m^T \{ \Sigma B^T + 2R\Sigma K^T + 2\Sigma PB \} \hat{e}_l .$$  \hfill (5.44)

Thus, enforcing the condition that Equation (5.44) is equal to zero for all \( \{l, m\} \), gives an equation for the optimal \( K \) in terms of \( \Sigma \) and \( P \) as

$$\Sigma B^T + 2R\Sigma K^T + 2\Sigma PB = 0 .$$  \hfill (5.45)

Multiplying the above equation from the left by \( \Sigma^{-1} \) and solving for \( K \) results in the standard linear energy harvesting gain matrix; i.e.,

$$K = -\frac{1}{R} B^T \left( P + \frac{1}{2} I \right) .$$  \hfill (5.46)

We do not need to take the partial derivative of \( H \) with respect to the Lagrange multiplier \( P \) as this will just give us back the constraint in Equation (5.30). Finally, we can substitute Equation (5.46) into Equations (5.30) and (5.42) to arrive at two simultaneous nonlinear algebraic equations for \( \Sigma \) and \( P \) that have to hold at the optimum; i.e.,

$$\left[ A - \frac{1}{R} BB^T \left( P + \frac{1}{2} I \right) + V \right] \Sigma + \Sigma \left[ A - \frac{1}{R} BB^T \left( P + \frac{1}{2} I \right) + V \right]^T + GG^T = 0$$  \hfill (5.47)

$$A^T P + PA - \frac{1}{R} \left( P + \frac{1}{2} I \right) BB^T \left( P + \frac{1}{2} I \right) + PV + V^T P - UPV - V^T PU^T = 0$$  \hfill (5.48)

where \( U \) and \( V \) are defined in Equations (5.38) and (5.39), respectively.

### 5.2.3 Iterative Algorithm

It is clear that Equations (5.47) and (5.48) are coupled nonlinear algebraic equations. As such, solutions for the stationary covariance matrix \( \Sigma \) and the Lagrange multiplier
\( P \) must be solved iteratively. Furthermore, we see that Equation (5.48) is a Riccati equation with extra linear terms. The addition of these terms requires that the solution for \( P \) be determined by solving a linear matrix inequality (LMI). Equation (5.48) can be equivalently represented as an LMI by taking its Schur complement; i.e.,

\[
\begin{bmatrix}
\bar{A}^T P + P \bar{A} - UPV - V^T PU^T & \bar{A}^T \left( P + \frac{1}{2} I \right) B \\
B^T \left( P + \frac{1}{2} I \right) & \bar{R}
\end{bmatrix} > 0 \tag{5.49}
\]

where \( \bar{A} = A + V \). The objective that \( P \) must satisfy in order to make the LMI in Equation (5.49) a convex linear programming problem is that we want to minimize

\[-\text{tr}\{ P [G G^T + 2V \Sigma U] \} \]

Thus, the solution for \( P \) can be obtained by solving the following optimization problem

\[ OP1 : \begin{cases} 
\text{Minimize:} & -\text{tr}\{ P [G G^T + 2V \Sigma U] \} \\
\text{Subject to:} & \text{Equation (5.49), } P = P^T < 0
\end{cases} \]

We now outline a simple iterative algorithm to solve for \( \Sigma \) and \( P \), consisting of the following steps for iteration \( i \).

1. Initialize \( \Sigma_0 \) and \( P_0 \) by solving the linear energy harvesting problem (i.e., with \( U = V = 0 \)).

2. Calculate \( U_i \) and \( V_i \) using \( \Sigma_{i-1} \).

3. Calculate \( P_i \) by solving the LMI optimization problem in \( OP1 \).

4. Calculate \( \Sigma_i \) by solving Equation (5.47).

5. Return to Step 2 with \( i \leftarrow i + 1 \) until convergence is reached.

Convergence of the algorithm is reached when the absolute value of the change in \( P_{gen} \) between the current and previous iteration is below a certain tolerance. The
value of $\bar{P}_{\text{gen}}$ at the first iteration can be calculated using Equation (3.43) while the value of $\bar{P}_{\text{gen}}$ at any other iteration can be calculated using (5.33).

In general, the algorithm presented here is able to converge to solutions for $\Sigma$ and $P$ within 10–20 iterations. However, if the ratio of the amplitude of the Coulomb friction divided by inertial force of the structure (i.e., the mass of the structure multiplied by the standard deviation of the disturbance acceleration) reaches a value that is too large, then the algorithm becomes numerically unstable. It is possible to augment the proposed algorithm such that the Coulomb friction amplitude is incrementally increased until it reaches a desired level. The theory presented in the previous subsection is valid for any level of Coulomb friction. For the example presented in the following section, the algorithm is numerically stable for the considered ratio of Coulomb friction to the inertial force of the structure.

5.2.4 Statistically Linearized Optimal Static Admittance

All of the prior analysis in this section has been for the case where we assume that the electronics are implementing the optimal state feedback controller. It is straightforward to extend this analysis to determine the optimal static admittance for an energy harvester with Coulomb friction introduced by the transducer. In this case, we have that the state feedback relationship $i(t) = Kx(t)$ is replaced by the relationship for the case where the electronics are implementing a static admittance; i.e., $i(t) = -Y_0B^T x(t)$, where $Y_0$ is the static admittance. Thus, the average power generated can be expressed by

$$\bar{P}_{\text{gen}} = (Y_0 - Y_0^2 R) B^T \Sigma B$$

(5.50)

where the covariance matrix must satisfy the following nonlinear algebraic equation

$$[A - Y_0 BB^T + V] \Sigma + \Sigma [A - Y_0 BB^T + V]^T + GG^T = 0$$

(5.51)
where the expression for $V$ remains unchanged from the previous subsection. Thus, we can determine the optimal $Y_0$ by solving the following optimization problem

$$\begin{align*}
OP2 : \quad & \text{Maximize:} \quad (Y_0 - Y_0^2 R) B^T \Sigma B \\
& \text{Over:} \quad Y_0 \\
& \text{Subject to:} \quad \text{Equation (5.51)}, \ Y_0 > 0
\end{align*}$$

Unlike the linear energy harvesting problem, we must iteratively update $\Sigma$ to solve for the optimal $Y_0$ that maximizes Equation (5.51). A similar iterative algorithm to the one proposed in the previous subsection can implemented to solve the optimization problem in $OP2$. It should be noted that for each iteration, the optimal $P_{gen}$ can be determined using a one-dimensional line search. In a similar manner to determining the optimal static admittance for the linear energy harvesting problem, one can employ a bisection algorithm to converge rapidly to the optimal $Y_0$, given $A$, $B$, $C$, $F$, $G$, $V$ and $R$. It was found that this algorithm is numerically stable for any ratio Coulomb friction to the inertial force of the structure.

5.3 Optimal Control of a SDOF Energy Harvester

5.3.1 Optimal Full-State Controller

Consider the SDOF oscillator and coupled electromagnetic transducer which is subjected to a stochastic disturbance $a(t)$ in Figure 5.4. For this example we assume

![Figure 5.4: A SDOF oscillator with an electromagnetic transducer which is implementing the optimal full-state feedback control law.](image)
that the electronics implement the full-state feedback relationship \( i_q(t) = Kx(t) \).

We also assume that the transducer introduces a Coulomb friction force into the dynamics of the SDOF oscillator. The dynamics of the ballscrew, bearings, and belt are neglected because it was shown that they have a negligible effect on the average power generated in the previous section. Thus, the equation of motion that describes this system is

\[
\ddot{m}\ddot{x}(t) + \ddot{c}\dot{x}(t) + \ddot{k}x(t) + F_f\text{sgn}(\dot{x}(t)) = ma(t) + c_ei_q(t)
\]  

(5.52)

where \( \ddot{m}, \ddot{k}, \) and \( F_f \) are defined in Equations (5.2)–(5.5), respectively. In addition, we assume the following definitions for \( \ddot{c} \) and \( c_e \):

\[
\ddot{c} = c + c_d
\]

(5.53)

\[
c_e = \frac{3K_e}{2l}.
\]

(5.54)

If we define the harvester state vector as

\[
x_h(t) = \begin{bmatrix} \sqrt{\ddot{k}} & 0 \\ 0 & \sqrt{\ddot{m}} \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}
\]

(5.55)

then the harvester dynamics can be expressed by the self-dual state space

\[
\frac{d}{dt}x_h(t) = A_hx_h(t) + B_hi_q(t) + G_ha(t) + F_h\text{sgn}(y(t))
\]

(5.56)

\[
v(t) = B_h^Tx_h(t)
\]

(5.57)

\[
y(t) = C_hx_h(t)
\]

(5.58)
where

\[
A_h = \begin{bmatrix}
0 & \sqrt{\frac{k}{\dot{m}}} \\
-\sqrt{\frac{k}{\dot{m}}} & -\frac{c}{\dot{m}}
\end{bmatrix},
\]  
(5.59)

\[
B_h = \begin{bmatrix}
0 \\
\frac{c_e}{\sqrt{\dot{m}}}
\end{bmatrix}^T,
\]  
(5.60)

\[
G_h = \begin{bmatrix}
0 \\
\frac{m}{\sqrt{\dot{m}}}
\end{bmatrix}^T,
\]  
(5.61)

\[
F_h = \begin{bmatrix}
0 \\
-F_f/\sqrt{\dot{m}}
\end{bmatrix}^T,
\]  
(5.62)

\[
C_h = \begin{bmatrix}
0 \\
1/\sqrt{\dot{m}}
\end{bmatrix}.
\]  
(5.63)

In addition, we have that the matrices that characterize the disturbance dynamics are defined in Equations (4.62)–(4.64). However, for this example, the (2,1) term in \(B_a\) is multiplied by the standard deviation of the disturbance acceleration; i.e.,

\[
B_a = \begin{bmatrix}
0 \\
2\sigma_a/\sqrt{\omega_a}
\end{bmatrix}^T.
\]  
(5.64)

Thus, we can formulate the augmented state space with the appropriate definitions for \(A\), \(B\), \(C\), \(F\), and \(G\). We set \(\omega_a = \sqrt{\frac{k}{\dot{m}}}, \sigma_a = A_0/\sqrt{2}\) (where \(A_0\) is the acceleration amplitude used in the example in Section 5.1), and \(R = R_c\) (i.e., the dissipative losses are equal to the coil resistance of the motor).

For this example, we begin by illustrating the convergence of the iterative algorithm that is proposed in the previous subsection. As such, we fix \(\zeta_0 = 0.2\) and \(\zeta_a = 0.5\) and run the algorithm with a convergence tolerance of \(10^{-6}\). In Figure 5.5, we see that the algorithm converges in 14 iterations to \(\bar{P}_{gen} = 2.17\)W. We compare this value to the performance of the frictionless energy harvester, which is \(\bar{P}_{gen} = 5.21\)W. As expected, including the nonlinear Coulomb friction force results in the average power generated being approximately 40% of the average power generated from a frictionless system. This gives us an idea of how much performance is sacrificed by having a transducer that introduces Coulomb friction into the dynamics of the SDOF
oscillator.

Next, we compare the average power generated by the frictionless energy harvester to the average power generated by the statistically linearized energy harvester over a range of $\zeta_a \in [0, 2]$. The plots in the Figure 5.6 illustrate these curves for passive viscous damping levels of $\zeta_0 = 0.15$, $\zeta_0 = 0.2$, and $\zeta_0 = 0.3$. We see that for all three levels of damping that the statistically linearized energy harvesting performance is approximately 40% of the value of the energy harvesting performance from the frictionless system.

5.3.2 Optimal Static Admittance

We now consider the case in which the electronics are implementing the optimal static admittance; i.e., $i_q(t) = Y_0 B^T x(t)$. The state space models describing the dynamics of the energy harvester and disturbance acceleration are exactly the same as those outlined in the previous subsection. As such, we can implement the iterative algorithm to solve for the optimal static admittance for a given value of $\zeta_a$. The plots in Figure 5.7 compare the average power generated with the electronics implementing the optimal static admittance for the frictionless system and statistically linearized system for $\zeta_0$ values of 0.15, 0.2, and 0.3. The average power generated resulting
from the statistically linearized is approximately 40% of the average power generated from the frictionless energy harvester, which is consistent with the optimal full-state controller. It is interesting to note that in the limit as $\zeta_a \to 0$ the optimal static admittance generates less average power than the optimal full-state feedback controller.

To illustrate the improvement in energy harvesting performance from a stochastic disturbance we plot the $\bar{P}_{gen}$ ratio for the statistically linearized SDOF energy harvesting system. Recall that this is the ratio of the average power generated with the electronics implementing the optimal static admittance over the average power generated with the electronics implementing the optimal full-state feedback controller.
Figure 5.7: Comparison of the average power generated by the frictionless and statistically linearized SDOF energy harvester for the optimal static admittance: (a) $\zeta_0 = 0.15$; (b) $\zeta_0 = 0.2$; (c) $\zeta_0 = 0.3$.

The $\bar{P}_{gen}$ ratio is plotted for the SDOF energy harvester in Figure 5.8 for $\zeta_0$ values of 0.05, 0.1, 0.15, 0.2, 0.25, and 0.3. From this plot, we see that the optimal full-state controller generates significantly more average power than the optimal static admittance over the entire range of $\zeta_a$ values. This observation is in contrast to the frictionless energy harvester in which the $\bar{P}_{gen}$ ratio approaches unity at the narrow-band limit. Finally, we can conclude that for an energy harvesting TMD system with low damping, it would be extremely beneficial in terms of average power generated to implement the optimal full-state controller instead of the optimal static admittance.
Figure 5.8: $\bar{P}_{\text{gen}}$ ratio for $\zeta_0$ values of 0.05, 0.1, 0.15, 0.2, 0.25, and 0.3 (from bottom to top).
Over the course of this thesis, we have developed a number of results. In the first section of this chapter, these results are summarized. Then, we will discuss some items of current and future research which relate to the ideas presented here.

6.1 Summary

In Chapter 2, we illustrate the design tradeoffs one faces, in the design of electromechanical transducers for use in large-scale vibratory energy harvesting applications. These design tradeoffs can be distilled down to a few criteria that should be used to balance various parameters of the machine, electronics, screw conversion, and structure, against each other. These criteria (or slight variations of them) hold with some generality, for applications of this technology in which the oscillation of the device is concentrated near a single frequency.

In addition to the design of the device, Chapter 2 compares two mechanical models that could be used to predict the response of the electromagnetic transducer. Both models include an expression for the high-frequency force oscillations that are attributable to the mechanics of the ballscrew. The main difference between the
models is related to the force element that is used to account for the hysteresis in the force-velocity plane. Based on an extensive literature search, it was found that one of the most common ways to model hysteresis is with a Bouc-Wen force element. For the Bouc-Wen model, we fit the parameters using experimental data in which the device was back-driven by a sinusoidal displacement at a single frequency. We show that this model is unable to accurately predict the response of the device when it is back-driven by a random displacement. Motivated by this fact, we propose a mechanical model that uses a nonlinear spring model to account for the hysteresis in the force-velocity plane. The nonlinear spring can be attributed to the elasticity of the belt that connects the ballscrew to the shaft of the synchronous machine. We show that back-driving the transducer with a sine-sweep can be used to determine the optimal parameters for the model using a Levenberg-Marquardt nonlinear least squares algorithm. Comparing the model with the optimal parameters to the experimental response of the device results in excellent agreement for a wide range of operating conditions, including a step change in load resistance, constant load resistance with random displacement, and random load resistance with random displacement.

In Chapter 3 we investigate the potential for enhanced energy harvesting performance from stochastic disturbances using the optimal full-state feedback controller, which is derived using LQG control theory. In order to formulate the optimal controller we begin by defining the general state space models that describe the harvester and disturbance dynamics. From these models, we derive the optimal state feedback controller that optimizes the average power generated from a stochastic disturbance. Next, we derive the optimal static and dynamic admittances for a nondimensional SDOF energy harvester with electromagnetic coupling. Finally, we illustrate the benefits of implementing the optimal state feedback controller over the optimal static admittance by plotting the ratio of the performance of these two controllers over a
range of disturbance bandwidths and for different levels of mechanical damping.

In Chapter 4, we present a theorem on the decoupling of the Riccati equation used to solve for the optimal energy harvesting controller. It turns out that if the state space system describing the augmented harvester and disturbance dynamics can be expressed as being self-dual, with the harvester dynamics being WSPR, then the Riccati equation decouples. This leads to an energy harvesting current relationship that only requires half of the states for feedback. For the nondimensional SDOF electromagnetic energy harvester, we show that the states required for feedback are the easiest to measure. As such, we present a passive analog circuit that could be used to implement the partial-state control law. Next, the improvement in performance using the partial-state gradient descent routine was investigated for a nondimensional TMD energy harvester. Despite the Riccati equation not decoupling, we illustrate the benefits in terms of the average power generated for including the disturbance acceleration in addition to voltage measurements in the energy harvesting current relationship. We show for various values of losses in the electronics, and for three levels of TMD mass, that more average power can be generated by optimizing the voltage and disturbance acceleration gains than by just optimizing a static admittance. The main result of this chapter is that the optimal partial-state feedback control law is able to significantly outperform the optimal static admittance.

Finally, Chapter 5 demonstrates the energy harvesting capability of the device for the case where the electromagnetic transducer is interfaced with a SDOF oscillator. In order to perform this analysis, we derive an analytical expression for the average power generated and use this expression to determine the optimal resistive load and optimal average power generated. Both the theoretical model using the experimentally fit parameters and the experimental system show similar results for the average power generated over the range of resistive loads, and for three levels of damping in the SDOF oscillator. Furthermore, for the case where the SDOF energy harvesting
system is subjected to a stochastic disturbance, we derive expressions for the stationary covariance matrix and the optimal full-state feedback controller using statistical linearization. The average power generated by the statistically linearized system is compared to the average power generated by the frictionless system for several levels of passive viscous damping. From these plots, we can conclude that implementing the optimal full-state feedback controller harvests significantly more average power than the optimal static admittance, including in the narrowband limit.

6.2 Future Work

The work presented in this thesis represents ongoing research. There are several areas which have the potential for further investigation. One possible area for future work would be to test the statistically linearized optimal full-state feedback controller and optimal static admittance using RTHT for a SDOF oscillator. Validation of the simulated results presented in Chapter 5 would be a crucial component to the development of energy harvesting from TMDs in tall buildings. Furthermore, incorporating a detailed loss model of the servo drive into the expression for the average power generated would allow us to optimize various properties of the electronics. Throughout this thesis we assumed that the servo drive was dissipating power like a simple resistor. However, the actual dissipative power losses in any power electronic drive are non-quadratic and a detailed model of the dynamics of the transistors in the servo drive would be required to obtain an accurate estimate of the losses. A non-quadratic loss model of the electronics could be used to optimize the bus voltage and switching frequency of the servo drive.

Another possible direction for this research would be on the development of a multi-objective structural control and energy harvesting controller that could be used to control TMDs in tall buildings. For the controllers presented in this paper, the performance objective that is being optimized is power extraction by the transducer.
However, the objective of passive TMDs in tall buildings is to reduce displacements and accelerations when the building is being excited by wind. It would be possible to replace the passive damping components of TMDs with active transducers, like the one described in this thesis, and control the TMD to reduce vibrations in the structure while also harvesting energy. The amount of power that could be harvested from such a system is still under investigation by several researchers. However, initial projections suggest that the power dissipated by TMDs in tall buildings is on the order of kW–MW (Ni et al., 2011).
Bibliography


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