Optimal causal control of a wave energy converter in a random sea

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ABSTRACT

This paper concerns the design of feedback control systems to maximize power generation of a wave energy converter (WEC) in a random sea. In the literature on WEC control, most of the proposed feedback controllers fall into three categories. Many are static; i.e., they extract power by imposing an equivalent damping or resistive load on the power take-off (PTO) devices. Others are dynamic and are designed to maximize power generation at all frequencies, which results in an anticausal feedback law. Other dynamic control design methods are causal, and are tuned to achieve the anticausal performance at only a single frequency. By contrast, this paper illustrates that the determination of the true optimal causal dynamic controller for a WEC can be found as the solution to a nonstandard linear quadratic Gaussian (LQG) optimal control problem. The theory assumes that the control system must make power generation decisions based on present and past measurements of the generator voltages and/or velocities. It is shown that unlike optimal anticausal control, optimal causal control requires knowledge of the stationary spectral characteristics of the random sea state. Additionally, it is shown that the efficiency of the generator factors into the feedback synthesis. The theory is illustrated on a linear dynamical model for a buoy-type WEC with significant resonant modes in surge and pitch, and equipped with three spatially-distributed generators.

1. Introduction

It is well-known that a substantial amount of energy exists in ocean waves, and the development of WEC systems clearly has an important role to play in the emergence of renewable energy technologies [1–4]. WECs built around the concept of an oscillating floating body, with an embedded PTO mechanism, arguably provide the most versatile and deployable approach to harvesting the resource. However, the complex dynamic coupling with the waves and the water, and the oscillatory nature of the available power, require the design of feedback control algorithms to optimize power generation. Optimization of WEC control systems constitutes a mathematical problem with certain well-known but challenging subtleties.

Typically, controllers for these WECs presume harmonic waves, and are designed according to the same network-theoretic impedance-matching principles used in the design and operation of antenna arrays and waveguides [5–7]. However, true sea states are stochastic, with power spectra usually characterized by Pierson–Moskowitz or JONSWAP spectra, which do not exhibit a particularly high quality factor [8]. For such cases, controllers derived via impedance-matching theory must impose a feedback law which is the Hermitian adjoint (i.e., complex-conjugate transpose) of the hydrodynamic impedance matrix for the WEC, at all frequencies. (For this reason, it is sometimes called “complex conjugate control,” as in [9,10].) Such controllers are always anticausal, and thus require some anticipatory technique in which present decisions are made with future wave information. This can be accomplished, for example, with the use of deployable wave elevation sensors. It can also be approximately implemented through the use of wave forecasting, using the dynamic response of the WEC and/or the surrounding water elevation to anticipate future wave excitation [11].

However, in many applications there may be considerable advantage to the use of feedback controllers that do not explicitly require future waves to be known (i.e., causal controllers), and which make power generation decisions based only on easy-to-measure feedback signals such as the generator velocities or their voltages. Although it is not yet a well-known fact in literature on WEC control, the causal controller that maximizes power generation from stochastic waves may be found as the solution to a nonstandard LQG optimal control problem. In this article, we investigate the design of such controllers, and examine their behavior and performance in relation to that of anticausal controllers.

Many prior studies have focused on optimality of feedback control systems for WECs, with the great majority of these studies focused on latching control [12], in which the power take-off system is alternately clamped and unclamped. Hoskin and Nichols formulated this as an optimal control problem, with the latching alternations modeled as an on/off control input, and used Pontryagin’s maximum principle to solve for optimal trajectories for the control inputs [13]. Numerous
subsequent investigations have extended this concept for latching control [14–17]. Additionally, tandem application of latching systems and active power flow control systems has also been investigated within this paradigm [18], as well as in the closely-related literature on model-predictive control (MPC) of WEC systems [19,20].

The present study is distinct from the aforementioned ones, in that it assumes the ocean wave excitation to be a stochastic disturbance which cannot be known ahead of time, and solves for the optimal causal feedback law under this information constraint. Beyond causality, the system is assumed to be unconstrained in the control inputs it can apply. Due to the fact that the optimal causal controller is within the LQG paradigm, it exhibits certainty-equivalence, and therefore may be broken into an observer subsystem and a state-feedback subsystem. The observer subsystem is a Kalman–Bucy filter that estimates the full dynamic state of the WEC system, as well as states used to model the dynamics of the surrounding fluid. The state-feedback controller then makes power generation decisions on these estimates. Because the optimal controller estimates the dynamic behavior of the waves, it requires knowledge of the stationary power spectrum for the sea state.

Although concepts discussed in this paper are applicable to a broad class of systems, they are illustrated in the context of the WEC in Fig. 1, comprised of a rigid buoy tied to three tethers attached through spools to generators at the sea floor. This “tripod” configuration is similar to the buoy analyzed by Srokosz [21]. A WEC prototype with many similar features is currently under development by Resolute Marine Energy, Inc. [22]. Note that although this WEC consists of only one rigid body, it has three generators, which may all be controlled independently, and due to the system geometry, it is capable of extracting power from surge, heave, and pitch motions simultaneously. As such, the optimization of power generation from this system constitutes a multivariable control problem.

This article makes a number of specific contributions, which extend results from recent conference papers by the authors [23,24]. First, for a generic, multi-generator WEC system with linear dynamics, we make and clarify the connections between optimal anticausal, optimal causal, and optimal static (i.e., damping) controllers for WECs. To accomplish this, the infinite-dimensional dynamics of a WEC must be approximated by a finite-dimensional state space. Next, in the context of the example WEC, we examine its stationary power flow (both real and reactive) with the static, anticausal, and causal optimal control designs. Several salient features of, and contrasts between, these controllers are illustrated via an analysis of the spectral characteristics of the optimized power flow. Finally, we conclude with a discussion regarding the sensitivity of optimal causal controllers to uncertainty in the spectrum and propagatory direction of the sea state.

1.1. Notational conventions

The notation in this article mostly follows standard conventions in system analysis. We denote \( j = \sqrt{-1} \), and denote \( \text{Re}(\cdot) \) and \( \text{Im}(\cdot) \) as the real and imaginary parts, respectively. For a time-domain vector \( q(t) \), the two-sided Laplace transform (assuming it exists) is \( \hat{q}(s) = \int_{-\infty}^{\infty} e^{-st} q(t) \, dt \). Assuming \( \int_{-\infty}^{\infty} q^2(t) |q(t)| \, dt < \infty \), the Fourier transform is then just \( \hat{q}(j\omega) \). All frequencies \( \omega \) are in rad/s. If a system maps \( p(t) \in \mathbb{R}^m \) into \( q(t) \in \mathbb{R}^n \), we say \( p \mapsto q \). If the system is linear then the input/output relationship is represented in the Laplace domain by a transfer matrix \( G(s) \in \mathbb{C}^{m \times n} \), with the associated frequency response (assuming \( G(s) \) is stable) evaluated in the complex phasor domain as \( G(j\omega) \). When it does not lead to ambiguity, we refer to such a system simply as \( G \). When such systems are finite-dimensional, we make use of the short-hand \( G \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) to imply that \( G(s) = D + C [ sI - A ]^{-1} B \). For a matrix \( Q \in \mathbb{C}^{m \times n} \), the notation \( Q \succ 0 \) implies positive-definiteness. For a square matrix \( Q \), \( \text{tr}(Q) \) is the trace. If \( \psi(Q) : \mathbb{R}^{m \times n} \to \mathbb{R} \) then the matrix derivative \( \partial \psi/Q \in \mathbb{R}^{m \times n} \) is just an element-by-element partial derivative. For linearization analysis, we denote the value of a vector \( q \) at a linearization point by \( q_0 \), and denote deviations by \( \delta q \). For stochastic analysis, \( \mathbb{E}(\cdot) \) denotes the expectation of a quantity. If \( q(t) \) is a stochastic response quantity, then the time-independent notation \( \mathbb{E}(\cdot) \) implies that the expectation is taken in stationarity.

2. System model

2.1. Mechanical dynamics

Fig. 1 illustrates a WEC system, consisting of a floating buoy (modeled as a rigid body), which is interfaced with three rotary generators through retractable tethers which spool around pulleys at the generator shafts. Each generator is anchored rigidly to the ocean floor, which is assumed to have constant depth. Tension is maintained for the equilibrium position through the use of springs, as shown. The location of the center of mass of the rigid body, relative to the origin, \( O \), of inertial reference frame, is the vector \( r \). Without loss of generality, we assume \( O \) is located at the equilibrium position of the buoy’s center of mass, such that in equilibrium, \( r = 0 \). Each retractable tether is mounted to the buoy via an ideal pin connection which is fixed on the buoy surface. In the figure, the vector pointing to the attachment point for tether \( i \), relative to its generator spool, is \( s_i \).

We first model the interaction of a single tether with the buoy, and to do this we will temporarily suppress superscripts denoting tether number (i.e., \( i = 1, 2, 3 \)). The tether attachment location on the body, relative to the center of mass, is \( b \). The tension in the tether is denoted \( t > 0 \). The applied force vector of the tether, on the buoy, is \( f = -e_b \), where \( e_b \) denotes the unit vector in the same direction as \( s_i \).

Consider the dynamics of the rigid body for small oscillations about equilibrium. The vector of angular displacements of the body \( \theta = \{ \theta_x, \theta_y, \theta_z \} \) is taken to be relative to the axes as shown, and these angles are assumed to be zero in equilibrium. The “small angle” approximation is also assumed, which allows for the order of angular rotations to be independent, and furthermore that \( \delta b \approx \theta \times b_0 = B_0 \theta \), where

\[
B_0 = \begin{bmatrix} 0 & b_{0z} & -b_{0y} \\ -b_{0z} & 0 & b_{0x} \\ b_{0y} & -b_{0x} & 0 \end{bmatrix}
\]  

(1)

the subscript “0” signifies equilibrium, and \( \delta b \) is the change in the vector \( b \) from \( b_0 \).

The change in \( s \) from its equilibrium value \( s_0 \) is \( \delta s = r + B_0 \theta \).
linearized change in the tether length is
\[ \delta \parallel s \approx e_{\parallel}^T \delta s, \]
where \( \parallel \) is the Euclidean norm. The change in \( f \) is
\[ \delta f = \delta \left( -\frac{t}{\parallel s \parallel} s \right) \]
\[ \approx -e_{\parallel} \delta t - \frac{t_0}{\parallel S_0 \parallel} (I - e_{\parallel} e_0^T) (r + B_0 \theta). \]

It is presumed that the variations in tether tension are a linear superposition of effects due to stiffness, viscous damping, and inertia of the transducer shaft (which are related to \( |s| \) and its derivatives) as well as a supplemental tension \( u \) due to the presence of an electromechanical torque. As such, it is also assumed that exist constants \( \{k, c, m\} \) such that
\[ \delta t = u + k \parallel s \parallel + c \frac{d}{dt} \delta \parallel s \parallel + m \frac{d^2}{dt^2} \delta \parallel s \parallel \]
\[ \approx u + e^T \left[ kr + cr + m r + B_0 (k \theta + c \dot{\theta} + m \ddot{\theta}) \right]. \]

The change in the moment \( h \) imparted on the rigid body, taken about the center of mass, is
\[ \delta h = \delta (b \times f) = b_0 \times \delta f - f_0 \times \delta b. \]

Defining \( S_0 \) similarly to \( B_0 \) and substituting in the expression for \( \delta f \) results in
\[ \delta h = \frac{t_0}{\parallel S_0 \parallel} B_0 (I - e_{\parallel} e_0^T) (r + B_0 \theta) + e^T \left[ kr + cr + m r + B_0 (k \theta + c \dot{\theta} + m \ddot{\theta}) \right]. \]

Defining \( \delta f \) and \( \delta h \) can be conveniently represented as
\[ \begin{bmatrix} \delta f \\ \delta h \end{bmatrix} = G_t \delta u - K_t \begin{bmatrix} r \\ \theta \end{bmatrix} - C_t \begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} - M_t \begin{bmatrix} r \\ \theta \end{bmatrix}, \]
where, using the fact that \( B_0 = -B_0^T \),
\[ G_t = -\begin{bmatrix} 1 \\ B_0^T \end{bmatrix} e_0 \]
\[ M_t = m G_t G_t^T \]
\[ C_t = c G_t G_t^T \]
\[ K_t = (k - \gamma_0) G_t G_t^T + \gamma_0 \begin{bmatrix} 1 \\ B_0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & S_0 B_0 \end{bmatrix}, \]
and where \( \gamma_0 = t_0 / \parallel S_0 \parallel \).

Now, consider a buoy with \( N \) such tethers. (I.e., for the example at hand, \( N = 3 \). The total dynamic component of the force and moment on the buoy is
\[ \begin{bmatrix} \delta f \\ \delta h \end{bmatrix} = \begin{bmatrix} \delta f_w \\ \delta h_w \end{bmatrix} + \begin{bmatrix} \delta f_{\parallel} \\ \delta h_{\parallel} \end{bmatrix} + \begin{bmatrix} \delta f_{\perp} \\ \delta h_{\perp} \end{bmatrix}, \]
\[ = \begin{bmatrix} \delta f_w \\ \delta h_w \end{bmatrix} - K_t \begin{bmatrix} r \\ \theta \end{bmatrix} - C_t \begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} - M_t \begin{bmatrix} r \\ \theta \end{bmatrix} - L^T u, \]
where \( f_w \) and \( h_w \) are the dynamic force and moment (respectively) due to fluid–structure interaction, \( u = \{u^1 \ldots u^N\}^T \), \( L = -\{G_1 \ldots G_N\}^T \), and \( K_t = \sum_{i=1}^{N} K_i \), with \( C_t \) and \( M_t \) defined similarly.

It can be shown that because \( b_j \) and \( s_j^0 \) constitute equilibrium positions \( \sum_{i=1}^{N} \gamma^j_0 S_i B_i^0 = 0 \). As such, the \( \gamma^j_0 S_i B_i^0 \) terms in each \( K_i \) vanish in the summation.

Regarding the wave forces and moments, they are presumed to be of the form
\[ \begin{bmatrix} \delta f_a \\ \delta h_a \end{bmatrix} = \begin{bmatrix} \delta f_{a,0} \\ \delta h_{a,0} \end{bmatrix} + \begin{bmatrix} \delta f_{a,b} \\ \delta h_{a,b} \end{bmatrix} + \begin{bmatrix} \delta f_{a,d} \\ \delta h_{a,d} \end{bmatrix}, \]
where \( \delta f_{a,0}, \delta h_{a,0} \) are the force and moment imparted on the buoy by the wave, \( \delta f_{a,b}, \delta h_{a,b} \) are the force and moment due to buoyancy, and \( \delta f_{a,d}, \delta h_{a,d} \) are the hydrodynamic (i.e., added mass and damping) forces.

Let the Fourier transforms of \( \delta f_a, \delta h_a \) be denoted \( \hat{\delta f}_a(j \omega) \) and \( \hat{\delta h}_a(j \omega) \). Then for the assumption of small deformations and linear wave theory, there is a \( 6 \times 1 \) transfer function matrix \( F_a(j \omega) \) relating the wave amplitude \( a \) to these forces and moments, i.e.,
\[ \begin{bmatrix} \delta f_a(j \omega) \\ \delta h_a(j \omega) \end{bmatrix} = F_a(j \omega) \hat{a}(j \omega). \]

Determining \( F_a(j \omega) \) involves solving a frequency-dependent series solution to the partial differential equation for a fluid–structure interaction problem, which can be approximately truncated to ensure a sufficiently high level of accuracy [25]. For reference, we provide the solution to \( F_a(j \omega) \) for an upright cylindrical buoy in Appendix A.

As is well known, the function \( F_a(j \omega) \) derived above is noncausal, because the wave amplitude and buoy response are both manifestations of a propagatory pressure, and the buoy response anticipates the arrival of the wave. In our analysis, the noncausality of \( F_a \) will be problematic, because it will prohibit us from approximating \( F_a \) by a finite-dimensional (causal) state space. Here, we rectify this by defining \( a \) as the wave amplitude at a distance of 5 m in front of the buoy, along the propagatory direction. This use of spatial delay is the same technique used by Falnes [26] to create causal approximations of \( F_a \), and was later used by Damaren [27] as an intermediate step in the finite-dimensional approximation of \( F_a \). As pointed out by both the above researchers, the delayed version of \( F_a \) is still noncausal. However, as Damaren also observed, it was found here that this approach led to very close approximation of \( F_a \) by a causal system.

For small deformations, the changes in buoyancy force \( \delta F_b \) and moment \( \delta h_b \) (relative to equilibrium) are linearly related to the displacements \( r \) and \( \theta \) via a buoyancy stiffness matrix \( K_b \), i.e.,
\[ \begin{bmatrix} \delta f_b \\ \delta h_b \end{bmatrix} = -K_b \begin{bmatrix} r \\ \theta \end{bmatrix}. \]

\( K_b \) is determined by linearizing the buoyancy stiffness forces about the static equilibrium position.

Let the Fourier transforms of \( \delta f_c, \delta h_c \) be denoted \( \hat{\delta f}_c \) and \( \hat{\delta h}_c \). Then for small deformations, these hydrodynamic forces are characterized by added-mass and added-damping relations, i.e.,
\[ \begin{bmatrix} \delta f_c \\ \delta h_c \end{bmatrix} = -\begin{bmatrix} 0 & M_c(\omega) + C_c(\omega) \\ 0 & J \end{bmatrix} \begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix}. \]

where \( M_c(\omega) \) and \( C_c(\omega) \) are the added mass and damping matrices respectively, and are also solvable via frequency-dependent series solutions to associated PDEs. As with \( F_a(j \omega) \), see Appendix A for the solutions to \( M_c(\omega) \) and \( C_c(\omega) \) for an upright cylindrical buoy, such as the one to be considered in the example later in the paper.

With the buoy forces found, the equation of motion is
\[ \begin{bmatrix} \mu I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \hat{\delta f}. \]

where \( \mu \) is the mass of the buoy and \( J \) is the rotational inertia matrix. Putting everything in the frequency domain and recognizing that \( \hat{\delta f} = \)
\( (j\omega)^{-1} \hat{\mathbf{r}} \) and \( \hat{\mathbf{v}} = j\omega \hat{\mathbf{r}} \) (and similarly for \( \hat{\theta} \) and \( \hat{\mathbf{\theta}} \)), the equating (21) to (16) gives
\[
\begin{bmatrix}
\hat{\mathbf{r}} \\
\hat{\theta}
\end{bmatrix} = \mathbf{W}^{-1}(j\omega) \begin{bmatrix}
\mathbf{F}(j\omega) \hat{\mathbf{\theta}} - \mathbf{L}^T \hat{\mathbf{u}}
\end{bmatrix}.
\]
(22)
where
\[
\mathbf{W}(j\omega) = j\omega \begin{bmatrix}
\mathbf{M}_s(a(o) + \mathbf{M}_r) \\
\mathbf{C}_s(a(o) + \mathbf{C}_r) + \frac{1}{j\omega} \left( \mathbf{K}_b + \mathbf{K}_r \right)
\end{bmatrix}
\]
(23)

2.2. Power conversion model

Let the tether extension velocity vector be
\[
\mathbf{z} = \left[ \frac{d}{dt} \mathbf{s} \parallel \cdots \cdots \frac{d}{dt} \mathbf{s}^N \right]^T.
\]
(24)
By reciprocity, we also have that for small deformations, these extension velocities are related to the buoy states by
\[
\mathbf{z} = \mathbf{L} \begin{bmatrix}
\hat{\mathbf{r}} \\
\hat{\theta}
\end{bmatrix}.
\]
(25)

We assume that the generators are three-phase permanent-magnet synchronous machines, each interfaced with a centralized DC power bus via a four-quadrant regenerative power-electronic drive. For this assumption, it is reasonable to assume that the three-phases are controlled such that the direct stator field (i.e., the component of the stator field parallel with the rotor field) is maintained at zero, while the quadrature field (i.e., the component of the stator field orthogonal to the rotor field) is explicitly controlled to regulate power conversion. This type of control maximizes the efficiency of the generators. It also simplifies the theoretical modeling of the generators, as it implies that there are effectively scalar quantities for internal voltage \( v' \) (i.e., the back-EMF) and corresponding current \( i' \) for each generator \( i \), which are of relevance to the system model. These are called the “quadrature components” of the three-phase voltage and current vectors.

We refer the reader to [28] for more detail on quadrature control of permanent-magnet machines. For the present paper, it suffices merely to state its implications for the system model. Assuming the generators to exhibit idealized linear behavior and minimal core losses, it results in linearity between extension velocity \( \mathbf{z}' \) and quadrature voltage \( v' \), for each generator \( i \), i.e.,
\[
\mathbf{v} = K_e \mathbf{z} = K_e \mathbf{L} \begin{bmatrix}
\hat{\mathbf{r}} \\
\hat{\theta}
\end{bmatrix},
\]
(26)
where we assume all generators to be identical, therefore possessing a uniform \( K_e \) value. By reciprocity, an equivalent linear relationship holds between the quadrature generator current vector \( \mathbf{i} \), and the resultant tether force vector \( \mathbf{u} \), i.e.,
\[
i = -\frac{1}{K_e} \mathbf{u}.
\]
(27)

Multiplying (22) through by the inverse of \( \mathbf{W} \) and multiplying by \( \mathbf{L} \) gives the transfer functions from \( \{a,d\} \) to \( \mathbf{v} \) as
\[
\dot{\mathbf{v}} = \mathbf{G}_d(j\omega) \hat{\mathbf{\theta}} + \mathbf{G}_i(j\omega) \hat{\mathbf{\theta}}.
\]
(28)
where
\[
\mathbf{G}_d(j\omega) = K_e \mathbf{L} \mathbf{W}^{-1}(j\omega) \mathbf{F}(j\omega),
\]
(29)
\[
\mathbf{G}_i(j\omega) = K_e^2 \mathbf{L} \mathbf{W}^{-1}(j\omega) \mathbf{L}^T.
\]
(30)

The electrical losses in the conversion process are assumed to be dominated by conductive dissipation. In reality, this dissipation is related to the current vector \( \mathbf{i} \) (and possibly the voltage vector \( \mathbf{v} \)) through a complex nonlinear relationship that involves many parameters of the electronic hardware (such as transistor and diode voltage drops, transition times, and gating energy), as well as operational parameters of the electronic drive, such as the DC bus voltage and switching frequency. However, for the purpose of mathematical tractability, we approximate the current–dependent losses as being resistive; i.e.,
\[
P_{\text{loss}} = \mathbf{R} \mathbf{i} \mathbf{i}.
\]
(31)
where \( \mathbf{R} \) is positive definite and diagonal. Each term on the diagonal of \( \mathbf{R} \) includes the stator coil resistance of the generator, together with an approximate transmission resistance for the drive.

It should be noted that the theory to be discussed here can be extended to accommodate more realistic electronic loss models. (See [29,30] for examples of this, as applied to smaller-scale vibratory energy harvesting problems.) However, these more realistic loss models do complicate the analysis, and are therefore not pursued here.

2.3. Sea state characterization

We assume \( a \) to be a stationary stochastic process with spectral density \( S_a(\omega) \), and with the normalization convention that the variance of \( a \) is
\[
\sigma_a^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_a(\omega) d\omega.
\]
(32)
where the frequency \( \omega \) is in rad/s. We assume \( S_a(\omega) \) is the standard JONSWAP power spectrum [8]. To review, this implies that \( S_a(\omega) \) is parametrized by its mean wave period \( T_1 \), significant wave height \( H_{1/3} \), and sharpness factor \( \gamma \), as
\[
S_a(\omega) = 310\pi \frac{H_{1/3}^3}{T_1^5 \omega^3} \text{exp} \left[ -\frac{944}{T_1^5 \omega^3} \right] \gamma^\phi.
\]
(33)
where
\[
\gamma = \text{exp} \left[ -\left(\frac{0.191}{\sqrt{2}\phi} - 1 \right)^2 \right]
\]
(34)
and
\[
\phi = \begin{cases} 0.07 : & \omega T_1 \leq 5.24 \\ 0.09 : & \omega T_1 > 5.24 \end{cases}
\]
(35)
The sharpness factor \( \gamma \) is constrained to be between \( 1 \) and \( 3.3, \gamma = 1 \) describing a fully developed sea.

2.4. General modeling assumptions

Although in this section we have developed a model for a specific type of WEC, it is important to note that the techniques discussed in the following sections apply more broadly, to general WEC models for which linear transfer matrices \( \mathbf{G}_a(j\omega) \) and \( \mathbf{G}_i(j\omega) \) can be found, and a resistance matrix \( \mathbf{R} \) is assumed. However, several mild assumptions do have to be made:

- \( \mathbf{G}_a(j\omega) \) and \( \mathbf{G}_i(j\omega) \) must be strictly proper, i.e.,
\[
\lim_{\omega \to \infty} \| \mathbf{G}_a(j\omega) \| = 0, \quad \lim_{\omega \to \infty} \| \mathbf{G}_i(j\omega) \| = 0
\]
(36)

In the case of \( \mathbf{G}_a(j\omega) \) this assumption is clearly justified, as it implies that the WEC has nonzero inertia. In the case of \( \mathbf{G}_i(j\omega) \), this assumption is also justified, because if it is not true for the originally-derived model, an equivalent model can be found for which, through re-formulation of \( \mathbf{R} \), the assumption is recovered.

- The wave elevation \( a(t) \) must be chosen to be at a location such that \( \mathbf{G}_a(j\omega) \) may be accurately approximated by a causal transfer function.
Finally, for $G_i(j\omega)$, we can assume without loss of generality that $G_i(j\omega)$ is positive-real; i.e.,
\[ G_i^T(j\omega) + G_i(-j\omega) \geq 0, \quad \omega \in [\infty, \infty] \]  
(37)
where the above inequality implies positive-semidefiniteness. This assumption holds on physical grounds, as it is equivalent to asserting that the WEC system is stable, and contains no non-conservative internal energy sources that influence its dynamics [31]. In Section 4.1, in which $G_i(j\omega)$ is approximated as a finite-dimensional state-space, this positive-real assumption must be strengthened to be “weakly-strict” [32]. In [33], it is shown that this stronger assumption is necessary in energy harvesting applications.

3. Optimal anticausal power generation

With the system model defined for the WEC buoy and a characterization for the stochastic ocean wave environment, the optimum power generation can be determined through classical impedance matching. It is assumed that the power electronics are controlled by a linear controller $Y$, which establishes an effective admittance between $v$ and $i$; i.e.,
\[ \hat{i}(s) = -Y(s) \hat{v}(s), \]  
(38)
The objective is to maximize the average (i.e., expected) total power generation, equal to the power extracted minus the losses; i.e.,
\[ P_{gm} = \mathcal{E}[-v^T i - i^T R] \]  
(39)
and $\mathcal{E}[\cdot]$ denotes the stationary expectation. $S_p(\omega)$ is the spectral density of generated power in the frequency domain, found as
\[ S_p(\omega) = G_i^H(1 + G_i Y) - H \Re \{Y \} - Y^H \mathcal{R} \{ Y \} [1 + G_i Y]^{-1} G_i S_a, \]  
(41)
where $(\cdot)^H$ denotes the Hermitian adjoint (i.e., complex-conjugate transpose), and $\Re \{ \cdot \}$ denotes the real component. The optimal anticausal controller in the frequency domain is found by first finding the $Y(j\omega)$ which maximizes $S_p(\omega)$ at each frequency, i.e.,
\[ Y(j\omega) = \left[ G_i^T(-j\omega) + 2R \right]^{-1}. \]  
(42)
The Laplace-domain representation of the optimal $Y(s)$ is just the analytic continuation of the controller above [34], i.e.,
\[ Y(s) = \left[ G_i^T(-s) + 2R \right]^{-1}. \]  
(43)
In [33], it was shown that when $G_i(s)$ and $G_i(s)$ are finite-dimensional and $Y(s)$ is weakly-strictly positive-real (WSPR), then the poles of $Y(s)$ are all in the open right-half plane and thus that its dynamics are always anticausal (i.e., anticipatory). As such, the controller makes decisions for power generation using only future behavior. Although the corresponding infinite-dimensional case is not discussed in [33], it is known that controllers for this case are also non-causal [35].

Note that at no point in the determination of the optimal anticausal controller, it is necessary to know $S_p(\omega)$. Rather, information about the sea state is only necessary to compute the optimal performance.

4. State space dynamic model

4.1. Finite dimensional approximation of WEC dynamics

In order to determine the optimal causal performance, we first approximate the infinite-dimensional system dynamics by finite-dimensional state spaces for $G_i$ and $G_o$ of the form
\[ G_o \simeq \begin{bmatrix} A_o & B_o & C_o \end{bmatrix}, \quad G_i \simeq \begin{bmatrix} A_i & B_i & C_i \end{bmatrix} \]  
(44)
where we assume the orders of these systems are $n_o$ and $n_i$, respectively. Customized techniques for accomplishing these finite-dimensional models for wave-excited systems have been proposed by Damaren [27,36]. However, here we accomplish the approximations using standardized subspace techniques originally proposed in [37], for determining finite-dimensional approximations for infinite-dimensional systems characterized by frequency response data. This is essentially the realization theory technique discussed in [38] for state-space modeling of WEC systems. Although this algorithm does not give optimal estimates of a given order, it does identify the balanced truncations of infinite dimensional systems in discrete-time from frequency-domain data. For convenience, Appendix B overviews the steps in this procedure, although no original contributions are claimed here regarding this method, and we refer the reader to [37] for proofs and more details. This identical procedure is followed to obtain finite-dimensional approximations for both $G_i$ and $G_o$.

Following this, the identified systems were augmented, as
\[ \begin{bmatrix} G_i & G_o \end{bmatrix} \sim \begin{bmatrix} A_i & B_i & C_i & 0 \end{bmatrix} \begin{bmatrix} 0 & B_o & C_o & 0 \end{bmatrix} \]  
(45)
Following this augmentation, a balanced truncation performed to remove redundancy in the dynamics of the two spaces, resulting in a standard state space for the harvesting system,
\[ G_{h} \triangleq [ G_i, G_o ] \sim \begin{bmatrix} A_h & B_h & E_h \end{bmatrix} \begin{bmatrix} C_h & 0 \end{bmatrix} \]  
(46)
in which $(A_h, B_h, E_h)$ is observable, and $(A_h, [B_h, E_h])$ is controllable. The order of the finite-dimensional approximate dynamic model above is denoted $n_h$, with $n_o = n_i = n_h$.

4.2. Recovering the WSPR property for the finite-dimensional model

The original infinite-dimensional model for $G_i(s)$ will adhere to the passivity conditions implied by the physics of the system, implying that it will be positive-real. However, passivity may be lost in the finite-dimensional approximation described in Section 4.1. It is vital that the approximate model be adjusted to recover this property, in order for the theory presented in the next section to hold. More specifically, the finite-dimensional model must be positive-real in the weakly-strict sense, which is a slightly more restrictive condition.

To review, $G_i(s)$ is positive real (PR) if (a) all its components are analytic for $\Re[s] > 0$, (b) all its components are real for positive real $s$, and (c) $G_i(s) + G_i^H(s) \geq 0$ for $\Re[s] > 0$, where $(\cdot)^H$ is the complex conjugate transpose [31]. It is Strictly Positive Real (SPR) if $G_i(s - \epsilon)$ is PR for some $\epsilon > 0$ [39]. Finally, $G_i(s)$ is WSPR, if the domain over which PR conditions (a)–(c) hold is extended to include $\Re[s] = 0$, and condition (c) is strengthened to a strict inequality [32]. As implied by the name, WSPR is more restrictive than PR, but less so than SPR.

To bring about the WSPR condition, parameters $(A_h, B_h, E_h, C_h)$ should be adjusted to obtain $(A_h, B_e, E_e, C_e)$. Ideally, they should be adjusted so as to minimize the change in $G_h$. In other words, if $G_i(s)$ is the identified finite-dimensional system and $G_i(s)$ is the WSPR adjusted system, then the error between the two should be minimal under some norm. One convenient norm is the $\mathcal{H}_\infty$ norm; i.e.,
\[ \| G_h - G_i \|_{\mathcal{H}_\infty} \leq \sup \sigma \left( C_h(j\omega) - G_h(j\omega) \right) \]  
(47)
where $\sigma(\cdot)$ denotes the maximum singular value.

In Appendix C, an algorithm is given, which adjusts $B_e$ only. It determines the $B_e$ which minimizes $\| G_h - G_i \|_{\mathcal{H}_\infty}$, constrained to the requirement that $G_i(s)$ be WSPR. The algorithm can be thought of as adjusting the zeros of $G_i(s)$ to recover the WSPR property, while leaving the poles unchanged, and it does not modify $G_h(s)$ at all. It is not truly optimal, because it only adjusts $B_e$, but not the other state space parameter matrices. However, it was observed to produce very accurate WSPR models.
To ease the notation, we will henceforth always assume the finite-dimensional model has been adjusted to be WSPR, and suppress the ($\gamma$) notation.

4.3. Finite-dimensional stochastic sea state model

Next, we find a finite-dimensional noise filter

$$F_w \sim \begin{bmatrix} A_w & B_w \\ C_w & 0 \end{bmatrix} \begin{bmatrix} 0 \\ C_w \end{bmatrix}$$

(48)

such that when excited by unit-intensity white noise, its power spectrum is close to the JONSWAP spectrum, i.e., $S_w(\omega) \approx |F_w(\omega)|^2$. Without loss of generality, we assume $\{A_w, B_w, C_w\}$ to be in controllable-canonical form; i.e.,

$$A_w = \begin{bmatrix} 0_{(nw-1)\times 1} & 1_{nw-1} \\ a_1 & \cdots & a_{nw} \end{bmatrix} \quad B_w = \begin{bmatrix} 0_{(nw-1)\times 1} \\ 1 \end{bmatrix}$$

(49)

$$C_w = [c_1 \cdots c_{nw}]$$

(50)

where $n_w$ is the order of the filter. The filter parameters $\{a_1, \ldots, a_{nw}, c_1, \ldots, c_{nw}\}$ should be such that $S_w(\omega) \approx |F_w(\omega)|^2$. Here, we choose the parameters which minimize the mean-square error $\int_{-\infty}^{\infty} (S_w(\omega) - |F_w(\omega)|^2)^2 d\omega$, while constraining the $a_k$ coefficients such that the system poles are in the open left-half plane. We also follow the simplified procedure advocated by Spanos [40], which sets $c_k = 0$ for all $k \neq 3$. It was found that an order $n_w = 4$ was sufficient to render close matching to JONSWAP spectra for reasonable sharpness factors. Fig. 2 illustrates the true spectrum and its fourth-order estimate for $\gamma = 1$.

4.4. System augmentation

The augmented finite-dimensional system, including the WEC dynamics and the stochastic wave filter model is

$$\begin{bmatrix} G_w \\ G_{xw} \end{bmatrix} \sim \begin{bmatrix} A & B & E \\ C & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ A_w \\ C_w \end{bmatrix}$$

(51)

where the augmented matrices $A$, $B$, $E$, and $C$ are

$$A = \begin{bmatrix} A_w & E \\ 0 & A_w \end{bmatrix}, \quad B = \begin{bmatrix} B \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ B_w \end{bmatrix}$$

(52)

$$C = \begin{bmatrix} C_h \\ 0 \end{bmatrix}$$

(53)

and the equivalent time-domain (i.e., state space) representation is

$$\dot{x}(t) = Ax(t) + Bi(t) + Ew(t)$$

(54)

$$v(t) = Cx(t)$$

(55)

where $w(t)$, the input to the noise filter, is white noise with unit intensity. This constitutes the dynamic system model we will use for the remainder of the paper.

5. Optimal causal power generation

5.1. Optimal power generation with full-state feedback

The objective is to find the full-state feedback law $K: \mathbf{x} \rightarrow \mathbf{i}$ which maximizes $P_{gen}$, or equivalently, which minimizes

$$-P_{gen} = \mathcal{E}[v^T \mathbf{i} + \mathbf{i}^T R \mathbf{i}]$$

(56)

This constitutes an LQG optimal control problem, but is non-standard because the performance functional is sign-indefinite. It is a standard result [41] that the optimal control solution, if it exists, is

$$i(t) = Kx(t)$$

(57)

$$0 = A^T P + PA - \frac{1}{2} \left[ B^T P + C \right]^T R^{-1} \left[ B^T P + C \right]$$

(58)

where $P$ is the stabilizing solution to the algebraic Riccati equation.

It has been shown [33] that the solution $P$ exists and feedback law (58) is stabilizing, if $G(s)$ is WSPR. Recall that this condition is assured for the infinite-dimensional problem by the physics of the problem, and is imposed on the finite-dimensional approximate model by the procedure in Section 4.2.

With full-state feedback law (58) implemented, the expression for the optimal causal power spectrum with full-state feedback is

$$S_p(\omega) = -\text{Re}[H^H(\omega) H_x(\omega)] - H^H(\omega) R H_x(\omega)$$

(59)

where

$$H_x(\omega) = K[j\omega I - A - BK]^{-1} E$$

(60)

$$H_x(\omega) = C[j\omega I - A - BK]^{-1} E$$

(61)

Furthermore, the closed-form expression for the optimal $P_{gen}$ can be related to $P$ as

$$P_{gen} = -\frac{1}{2} E^T PE$$

(62)

See [33] for a proof that for WSPR systems, (63) is always positive.

5.2. Optimal power generation with voltage feedback

In practice, the state vector $x$ will need to be estimated from sensor measurements, with the optimal estimates determined via a Kalman–Bucy filter. Although power generation controllers could be developed assuming arbitrary sensor outputs, we will here restrict our attention to the case in which the only sensor measurements are the generator voltages, $v(t)$, or equivalently, the generator velocities. The running state estimate $\chi(t)$ evolves according to

$$\dot{\chi}(t) = A\chi(t) + Bi(t) + F(C\chi(t) - v(t))$$

(63)

where $F$ is the observer gain. The controlled currents are then found through the certainty-equivalence principle, as

$$i(t) = K\chi(t)$$

(64)
with $K$ computed as before. Assuming white noise in the measurement channels for $v(t)$, with spectral intensity matrix $\sigma^2 I$, the optimal estimates are thus obtained via the standard Kalman observer gain; i.e.,

$$F = -\frac{1}{\nu}SC^T,$$

(66)

where $S$ is the solution to the matrix Riccati equation

$$0 = AS + SA^T - \frac{1}{\nu}SC^TCS + EE^T.$$  

(67)

The resultant optimal controller is a dynamic voltage feedback law

$$\dot{y}(t) = K \{ sI - A - BK - FC \}^{-1} F.$$  

(68)

Note that unlike the anticausal controller, the optimal causal controller depends on the wave spectrum.

The augmented closed-loop system dynamics are then described by the augmented state space $\dot{x}(t) = \{ x^T(t) \dot{x}^T(t) \}^T$ in which the stochastic dynamics governed by

$$\dot{x}(t) = \mathscr{A}x(t) + E \left[ \begin{array}{c} w(t) \\ \nu^{-1/2} n(t) \end{array} \right]$$  

(69)

$$v(t) = Cx(t)$$  

(70)

where $n(t)$ is the vector of sensor noises (which are each modeled as white, uncorrelated noise with spectral intensity $\nu$), and where

$$\mathcal{A} = \begin{bmatrix} \mathcal{A} & \mathcal{B}K \\ -\mathcal{F}C & \mathcal{A} + \mathcal{F}C + \mathcal{B}K \end{bmatrix}, \quad E = \begin{bmatrix} E & 0 \\ 0 & -F \sqrt{\nu} \end{bmatrix}. $$  

(71)

$$C = \begin{bmatrix} C & 0 \end{bmatrix}. $$  

(72)

The causal-optimal power generation spectrum is then given by

$$S_p(\omega) = -\text{tr} \left\{ \text{Re} \left[ H^H_{v}(j\omega) H_{v}(j\omega) \right] + H^H_{i}(j\omega) RH_{i}(j\omega) \right\},$$  

(73)

where in this case the transfer functions $H_v$ and $H_i$ are

$$H_v(j\omega) = K(j\omega I - \mathcal{A})^{-1} E,$$  

(74)

$$H_i(j\omega) = C(j\omega I - \mathcal{A})^{-1} E,$$  

(75)

and

$$K = \begin{bmatrix} 0 & K \end{bmatrix}. $$  

(76)

The optimal power generated is

$$\bar{F}_{gen} = -\frac{1}{\nu}EE^T PE - \text{tr} \left\{ KSK^T R \right\}. $$  

(77)

It is interesting to note that in the equation above, the inability to accurately measure the output voltage impedes the ability of the WEC to generate power. In [33], the asymptotic case as $\nu \to 0$ is investigated. If $G_e(s)$ is minimum-phase, this asymptotic case converges to the full-state performance, i.e., (63). When it is not, the asymptotic limit in the expression above is lower, by a finite amount. However, it can be proved that irrespective of the value of $\nu$, or the particular parameters of the problem, (77) is always positive.

It is also worth noting that (77) is the optimal power generation over all causal feedback laws, linear as well as nonlinear. Of course, this statement is predicated on our assumptions of linear dynamics, stationary stochastic response, and resistive transmission losses. If any of these assumptions are violated, there may be nonlinear controllers that outperform the best linear controller.

### 5.3. Optimal power generation with static damping

As a point of comparison, we note that it is a common practice in many energy harvesting applications, to regulate power extraction in a manner which emulates the imposition of resistive shunts across the generator terminals (or equivalently, viscous dampers in place of the PTO systems). In the context of our development above, this constitutes the imposition of a decentralized, static feedback law

$$i(t) = -Yv(t)$$  

(78)

where $Y$ is the effective static admittance matrix for the generators, which we assume to be diagonal; i.e.,

$$Y = \text{diag} \{ \ldots, Y_k, \ldots \}.$$  

(79)

The static values of $Y_k$ thus become the design parameters for the control system. The closed-loop system has the state space

$$x(t) = [A - BYC]x(t) + EW(t),$$  

(80)

$$v(t) = CX(t)$$  

(81)

In stationary stochastic response, the covariance matrix $S = \sigma [xx^T]$ is computed as the solution to the Lyapunov equation

$$0 = [A - BYC] S + S [A - BYC]^T + EE^T$$  

(82)

and the average power generation is

$$\bar{F}_{gen} = tr \left\{ (Y - Y^T RY) C S C^T \right\}.$$  

(83)

The problem thus becomes the algebraic optimization (maximization) of $\bar{F}_{gen}$ over the domain of diagonal $Y$ with diagonal components $Y_k > 0$. This optimization can be straightforwardly accomplished via a gradient ascent algorithm. Denoting $\frac{\partial \bar{F}_{gen}}{\partial Y}$ as the matrix gradient of $\bar{F}_{gen}$ with respect to each component of $Y$, we have that this gradient can readily be computed as

$$\frac{\partial \bar{F}_{gen}}{\partial Y} = -2BYPSC^T - (1 - 2RY) C SC^T,$$  

(84)

where $P$ is the solution to the dual Lyapunov equation

$$0 = [A - BYC]^T P + P [A - BYC] + C^T (Y^T RY - \frac{1}{2}Y - \frac{1}{2}Y^T) C.$$  

(85)

The gradient of $\bar{F}_{gen}$ with respect to each $Y_k$ term is found as the corresponding diagonal component of (84). The determination of the optimum $Y$ may thus be accomplished via any of several standard gradient-based algorithms, such as the conjugate gradient method. The frequency spectrum of generated power, $S_p(\omega)$, is then found as in (41), but with the static matrix $Y$ as found above.

### 6. Example

We consider the cylindrical buoy in Fig. 3. The three tethers extend from points on the buoy 2.25 m above the buoy’s bottom rim and are separated by 120° angles in the horizontal plane with one tether extending along the positive $x$-axis. All tethers point toward the centroid of the buoy.

Fig. 4 shows the exact (infinite-dimensional) transfer matrices $G_e(j\omega)$ and $G_0(j\omega)$, along with the finite dimensional approximations. These plots clearly show three modes of resonance. The first mode is $G_e(j\omega)$, the wave amplitude $a$ is taken to be the wave amplitude 5 m ahead of the buoy in the propagatory direction. Throughout the example, we assume $R = R$. Rather than parametrizing results in terms of $R$ directly, instead it will be convenient to refer to the short-circuit damping, $C_e$, of the generators, defined as

$$C_e = \frac{K e}{R}.$$  

(86)
6.1. Optimal power generation as a function of sea state

Fig. 5 shows optimal $\mathcal{P}_{\text{gen}}$ values for anticausal, causal, and static feedback, for three values of $C_e$ and as a function of $T_i$. The dark lines in the plot are for the case in which the performance for each $T_i$ is optimized for that particular sea state. Fig. 6 shows corresponding ratios of optimal $\mathcal{P}_{\text{gen}}$ values for the causal and static cases, each normalized by the anticausal case. As expected, both optimal causal and static controllers do worse than the anticausal case, with the distinction between the three becoming smaller as $C_e$ is made smaller. Contrasting the left and right columns of Fig. 5, we see the influence the sharpness factor $\gamma$ has on the available power. Energy in seas states with higher sharpness factor is more concentrated near a single frequency, and consequently the WEC can generate more energy from them through exploitation of resonance.

6.2. Spectral analysis of generated power

Fig. 7 shows the anticausal and causal optimal power spectra for the example buoy provided for various values of $T_i$ and $C_e$. Note that for large enough values of $C_e$ corresponding to more efficient generators, the optimal causal controller injects power into the system over a large range of frequencies, as is evident by the negativity of $S_\alpha(\omega)$ for this case. Meanwhile, as $C_e$ is made lower, both the causal and anticausal controllers are close to the static case, with all three having very low power spectra except near the resonant frequencies.

The buoy’s two lower resonant modes, surge and pitch, were designed so as to bracket frequencies corresponding to reasonable values of mean wave period $T_1$. As can be seen in the plots, for high values of $C_e$, this strategy makes it possible for the causal controller to extract power effectively, not just at these two frequencies, but at all frequencies in between. As such, the WEC is effective in broadband stochastic power generation with causal control, assuming the spectrum of $S_\alpha(\omega)$ is known.

Beyond the evaluation of the actual power generated by the optimally-controlled WEC, and its associated spectrum, it is also important to examine its reactive power flow. This is comprised of the oscillations of power in and out of each generator, which have zero average and thus generate no net energy, and yet are necessary to favorably influence the WEC’s dynamics to absorb more energy from the ocean. In a broadband stochastic context considered here, we examine the effective power factor of the power generated at each frequency. As with the traditional notion of power factor (for sinusoidal oscillations), a power factor close to unity implies that a generator’s voltage and current are approximately in phase, and that power flows most of the time from the mechanical system into the electrical system. A power factor close to zero implies that a generator’s voltage and current are approximately a quarter-cycle out of phase, implying that the majority of the power extracted by the generator is returned to the mechanical system. A power factor close to $-1$ implies that a generator’s voltage and current are in phase but with opposite sign, implying that power flows mostly from the electrical system into the mechanical system.

To define a stochastic power factor, we define the voltages $\mathbf{v}'$ for each generator $i$, as $\mathbf{v}' = \mathbf{v} + \mathbf{R}\mathbf{i}$, where $\mathbf{R}$ is the load, $\mathbf{H}$ is the transfer function, and $\mathbf{v}$ is the voltage. We then have that the stationary power spectra for $i'$ and $i''$ are just

$$S_{\mathbf{v}_i}(\omega) = \mathbf{H}_i(\omega) (\mathbf{H}_i(\omega))^T$$

and (the complex-valued) cross-spectrum between $i'$ and $i''$ is

$$S_{\mathbf{v}_i}(\omega) = \mathbf{H}_i(\omega) (\mathbf{H}_i(\omega))^T$$

As a general rule of thumb, the value of $C_e$ increases with the size, efficiency, and quality of the machines used for transduction. It is therefore economical to design a WEC to possess the lowest value of $C_e$ that still gives acceptable performance.

For the control design, the measurement noise $\nu$ was taken to be equal to $10^{-2} \mathrm{V}^2\mathrm{s}$. This value is very small, compared to typical stationary power spectra for $\nu$, and thus the results presented here can be viewed as being near the limiting case as $\nu \to 0$, corresponding to extremely accurate feedback measurements.
Fig. 5. $P_{\text{gen}}(T_1)$ for optimal anti-causal control (thick solid), optimal causal control (thick dash), and optimal static control (thick dot), optimized for each $T_1$ (i.e., gain-scheduled). Performance of non-gain-scheduled causal and static controllers optimized for $T_1 = 5\,\text{s}$, $7\,\text{s}$, and $10\,\text{s}$ are shown as thin lines. $C_e$ and $\gamma$ values are as shown for the six plots.

Fig. 6. Ratio of optimal $P_{\text{gen}}(T_1)$ values for causal over anti-causal (solid) and static over anti-causal (dashed). Generator efficiencies are $C_e = 10^5 \,\text{kg/s}$ (top), $10^4 \,\text{kg/s}$ (middle), and $10^3 \,\text{kg/s}$ (bottom). Sea state is characterized by $\gamma = 1$ (left) and $3.3$ (right).
power factor spectrum for generator \( i \) as

\[
p_i(\omega) = \frac{-\text{Re}\{S_{ei}(\omega)\}}{\sqrt{S_{ee}(\omega)S_{ii}(\omega)}}
\]  

(89)

Fig. 8 shows these power factors for the example at hand, for all generators. (Generators 2 and 3 have identical controllers and responses, due to the symmetry of the problem.) Power factors are given for the cases of anticausal control, causal control, and static control. However, note that the static case is trivial, because the constraints of the problem always result in a power factor of unity for all generators and at all frequencies. For the anticausal case, recall from Fig. 7 that the power generation spectrum \( S_{p}(\omega) \) is positive at every frequency, implying that in total, the three generators always extract more energy than they deliver to the WEC. However, it is clear from Fig. 8 that this is not true for each generator, taken individually. Indeed, for large values of \( C_e \), we see that at the surge resonance, generator 1 has a power factor close to 1, while the other two generators have a power factor close to \(-1\). This implies that in this frequency band, generator 1 is delivering power to the electrical system, whereas generators 2 and 3 are actuating the WEC. Meanwhile, at the pitch resonance, the opposite is true. Qualitatively, similar observations can be made about the optimal causal case, although the two cases have distinct behaviors. Also, note that for both the anticausal and causal cases, as \( C_e \) becomes larger, the power factors away from resonance tend to gravitate toward zero, implying that if the generators are highly efficient, there is significant benefit to the use of reactive power flow to harvest power at non-resonant frequencies.

6.3. Sensitivity to uncertainty in mean wave period

In realistic wave energy applications, the spectral content of a sea state will vary with time, and consequently, there is some advantage to the design of gain-scheduled controllers that adapt to these changes. Indeed, an optimal causal controller which is designed for optimal performance in one sea state can perform rather poorly in another, and can even operate at negative efficiency (i.e., it can result in net power loss rather than generation). The sensitivity of power generation to \( T_1 \) is also shown in Fig. 5. The light lines in the plot constitute the design of a controller for anticipated \( T_1 \) values of 5, 7, and 10 s, but subjected to the full range of values from \( T_1 = 4-12 \) s. For optimal causal controllers, the sensitivity of \( P_{gen} \) is more enhanced as \( C_e \) becomes larger. Note that for the case with \( C_e = 10^5 \) kg/s, negative efficiencies are clearly seen, when \( T_1 \) is far from the design value. For optimal static controllers, \( P_{gen} \) never becomes negative irrespective of how much \( T_1 \) is varied from its design value (which is to be expected) but performance is nonetheless significantly hampered when \( C_e \) is large.

6.4. Sensitivity to uncertainty in propagatory direction

It is an interesting fact that for the tripod-shaped WEC under consideration, if all the generators are identical (i.e., if their \( K_e \) and \( R \) values are the same) then the optimal causal and anticausal power generation \( P_{gen} \) is always the same, irrespective of the rotational orientation of the tripod (or, equivalently, the propagatory direction of the wave), assuming this direction is known and designed for. This fact arises due to symmetry, and is true even though the optimal controller \( Y(s) \) that actually attains the optimal performance does vary with the propagatory direction. As the direction changes, the three generators participate in varying ways and to varying degrees, to achieve the optimal power generation.

To see why this is the case, suppose the orientation of the tripod formed by the three tethers is rotated by some angle \( \beta \) in the \( x-y \) plane. Then it turns out that the dynamic characterization for the buoy is the same, except that \( L_0 \) must be modified. Let \( L_\beta \) denote the
value of \( I \) for a given rotation angle \( \beta \). Then it is a fact that
\[
L_\beta = \Theta(\beta) L_0
\]
where \( \Theta(\beta) \) is equal to
\[
\Theta(\beta) = \begin{bmatrix}
\Theta_{11}(\beta) & \Theta_{12}(\beta) & \Theta_{13}(\beta) \\
\Theta_{12}(\beta) & \Theta_{13}(\beta) & \Theta_{12}(\beta) \\
\Theta_{13}(\beta) & \Theta_{12}(\beta) & \Theta_{11}(\beta)
\end{bmatrix}
\]
where
\[
\Theta_{11}(\beta) = \frac{1}{6} (4 \cos \beta + 2)
\]
\[
\Theta_{12}(\beta) = \frac{1}{6} (-2 \cos \beta - 2 \sqrt{3} \sin \beta + 2)
\]
\[
\Theta_{13}(\beta) = \frac{1}{6} (-2 \cos \beta + 2 \sqrt{3} \sin \beta + 2)
\]
This is a (unitary) rotation matrix, and as such, has following properties for all \( \beta \):
\[
\Theta^T(\beta) = \Theta^{-1}(\beta) = \Theta(-\beta)
\]

Now, for a given wave excitation record \( a(t) \) and angle \( \beta = 0 \), denote the voltage vector as \( v_0(t) \) and the current vector as \( i_0(t) \). Then the equivalent voltage and current vectors for \( \beta \neq 0 \) (i.e., the ones for which the buoy dynamics are identical) are
\[
v_\beta(t) = \Theta(\beta) v_0(t)
\]
\[
i_\beta(t) = \Theta(\beta) i_0(t)
\]
Now, if the generators are all the same, then \( R = RL \), and the instantaneous power generated for this equivalent system is
\[
P_{\text{gen}}(t) = v_\beta^T(t) i_\beta(t) - i_\beta^T(t) R i_\beta(t)
\]
\[
= (\Theta(\beta) v_0(t))^T (\Theta(\beta) i_0(t)) + R (\Theta(\beta) i_0(t))^T (\Theta(\beta) i_0(t))
\]
\[
= v_0^T(t) \Theta^T(\beta) \Theta(\beta) i_0(t) - R i_0^T(t) \Theta^T(\beta) \Theta(\beta) i_0(t)
\]
\[
= v_0^T(t) i_0(t) - R i_0^T(t) i_0(t)
\]
Therefore, it may be said that for the WEC orientation with \( \beta = 0 \) and with currents \( i_0(t) \), there exists an equivalent set of currents \( i_\beta(t) \) for any rotational angle \( \beta \), such that the buoy dynamics and the generated power are both identical to the \( \beta = 0 \) case. Let \( Y_\beta(s) \) be the optimal controller for rotation angle \( \beta \). Then because of the observations above, it is straightforward to show that the optimal controller for this \( \beta \) is therefore related to the one with \( \beta = 0 \), through
\[
Y_\beta(s) = \Theta(\beta) Y_0(s) \Theta(-\beta)
\]
We conclude that all power generation plots are exactly the same as they would be for any rotational orientation (or, equivalently, any propagatory direction).

Consequently, the power generation plots in Fig. 5 are actually independent of the rotational orientation of the tripod, assuming the propagatory direction of the waves is known with certainty. However, if \( \beta \) is unknown and its value merely assumed in control design, errors in this quantity will lead to a departure from the optimal \( P_{\text{gen}} \) for the optimal causal controller. Thus, we close this example with an examination of the robustness of optimal causal control to errors in \( \beta \).

Fig. 9 shows a ratio of \( P_{\text{gen}} \) values for optimal causal controllers, as a function of \( \beta \). The denominator in the ratio is the power generation for a controller optimized for the \( \beta \) and \( T_1 \) values given, which, as we have shown, is invariant on \( \beta \). Meanwhile, the numerator in the ratio is the power generation for a controller optimized for the \( T_1 \) value given and for \( \beta = 0 \), but subjected to the \( \beta \) value given. As such, the ratio shows the degradation in power generation performance, as a consequence of an error in the assumed value of \( \beta \). Ratios are given for six combinations of \( C_e \) and \( y \) values. As can be seen, the optimal causal controller is highly sensitive to propagatory direction. For errors less than about 20°, performance is within 80% of the theoretical optimum, but this percentage drops fast after this, and even at 35°, power generation is zero for the highly efficient system (\( C_e = 10^5 \) kg/s). For efficient conversion, larger errors in \( \beta \) not only lead to negative efficiency (i.e., net energy escaping from the WEC into the sea) but astronomically negative efficiencies as extreme as \(-800\%\).

This illustrates that although optimal causal control can be advantageous, due to the fact that it does not require preview information about future waves, it does require accurate information about the sea state in order to be effective. This suggests that optimal causal control is only a viable approach if the controller parameters are scheduled so as to adapt to changes in the sea state, or in applications where the sea state parameters are highly predictable.
Fig. 9. Ratios of causal power generation given $T_1$ for the case where the propagatory direction is unknown and off by an amount $\beta$, over the case where propagatory direction is known accurately. The six plots are for $C_e = 10^3$, $10^4$, and $10^5$ kg/s (top to bottom) and $\gamma = 1$ and 3.3 (left to right).

7. Conclusions

The primary objective of this paper has been to illustrate that in WEC control applications for which wave preview sensing is either undesirable or impossible and the sea state is stochastic, optimal causal controllers can be designed using LQG optimal control theory. We have made comparisons of these optimal causal controllers to the anticausal and static cases, both in terms of their overall power generation performance, as well as their spectral characteristics. Finally we have shown that for optimal causal controllers, it is necessary that they be gain-scheduled in accordance with changes in the spectral content and propagatory direction of the sea state, in order to maintain acceptable performance.

With the recognition that causal WEC control problems fall into the LQG control paradigm, the wealth of knowledge on extensions of LQG controllers can readily be investigated for such problems. One immediate example of this concerns the application of various techniques for accommodating nonlinearities, such as saturation, into the model and control design. One straight-forward, if approximate, technique for accomplishing this involves the use of statistical linearization to account for nonlinearities in the optimization of linear feedback laws. This has been applied to other energy harvesting problems [30]. Additionally, many techniques for robust control (i.e., $H_\infty$ and $\mu$-synthesis techniques [42]) appear to be directly transferable to this problem, and may provide a systematic way of dealing with model uncertainty in WEC control design. Finally, these techniques can be framed in an indirect adaptive control context, in which the optimal LQG controller is continually adapted to an updated identified model of the system (including the sea state), based on measured output data. This would be analogous to adaptive tuning techniques proposed in prior studies (e.g., [43]), in which control parameters are modified in response to detected changes in wave period and amplitude.
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Appendix A. 6-DOF hydrodynamic coefficients for an upright cylindrical buoy

We consider an upright cylindrical buoy as in Fig. 3. Let the water depth be $d$, and the distance from the base of the cylinder to the ocean floor be $h$. For the example, $d = 20 \text{ m}$ and $h = 17 \text{ m}$). Without loss of generality, we can assume the reference frame is rotated such that the propagatory direction of the incident wave is coincident with the $x$-axis. As such, the physical interpretation of degrees of freedom $i = 1$...6 are: $i = 1$: surge, $i = 2$: sway, $i = 3$: heave, $i = 4$: roll and $i = 5$: pitch and $i = 6$: yaw.

The solution for the wave and hydrodynamic forces on the buoy can be expressed by an infinite-series representation involving Bessel-functions as eigenfunctions [25]. In that paper, it is explained that the potential field for the fluid region adjacent to the cylinder, and the potential field beneath the cylinder, are both expressed by different infinite-series and then summed up to calculate final forces. Subscripts $m$ and $n$ are used to denote the series orders in these respective fluid regions. Due to the revolution-symmetry only $m \in \{0,1\}$ contributes to the solution for a cylindrical buoy, whereas $n \in \{1...N_n\}$ where $N_n$ is the order of truncation (chosen as 25 in this study).

We begin by calculating some intermediate variables which appear in all hydrodynamic coefficients. Let $I_m(\cdot)$ and $K_m(\cdot)$ denote the modified Bessel functions of first and second kind, respectively, each corresponding to integer value $m$. We denote $a_m$, $a \in \{2...N_n\}$ as the lowest $N_n$ real solutions (chosen as 10 in this study) of the transcendental equation

$$\omega^2 + a_m \tan(\omega d) = 0 \quad (A.1)$$

and denote the unique imaginary solution to the above as $a_1 = -jk$, where $k > 0$ is thus the solution to the dispersion equation. Next, define $Q_m \in \mathbb{R}^{N_n \times N_n}$ element-by-element as

$$Q_{nn} = (-1)^n \frac{a_m h}{\lambda_m} \sin(\omega a_m) \quad (A.2)$$

where $\lambda_m = 2^{1/2}(1 + \sin(\omega a_d)/(2a_d))^{-1/2}$, and define vectors $\lambda_1 \in \mathbb{R}^{N_n}$, $\lambda_2 \in \mathbb{R}^{N_n}$, $\lambda_3 \in \mathbb{R}^{N_n}$, $\lambda_4 \in \mathbb{R}^{N_n}$ and $\lambda_5 \in \mathbb{R}^{N_n}$ element-by-element as

$$\lambda_{1m} = \frac{b_{a_m}}{a_m} [\sin(\omega a_m) - \sin(\omega a_d)] \quad (A.3)$$

$$\lambda_{3n} = \frac{b^{3/2} h^{2/3} l_2 ((n-1)\pi b/h)}{(n-1)\pi l_1 ((n-1)\pi b/h)} \cos((n-1)\pi) : n \geq 2 \quad (A.4)$$

$$\lambda_{4n} = \frac{b^3 h}{12a_d} l_2 ((n-1)\pi b/h) \cos((n-1)\pi) : n \geq 2 \quad (A.5)$$

$$\lambda_{5a} = \frac{b_{a_d}}{a_d} \left[\cos(\omega a_d) - \cos(\omega a_1) + (d - e) \sin(\omega a_d) - (h - e) \sin(\omega a_1)\right] \quad (A.6)$$

where $b$ is the radius of the buoy and $e$ is the position of the point with respect to which the moments are calculated (taken to be the geometric centroid of the buoy for this example). For each $m$, define vectors $w_m \in \mathbb{R}^{N_n}$ and $c_m \in \mathbb{R}^{N_n}$ element-by-element as

$$w_m = m - a_m b \frac{K_{m+1}(\omega a_d)}{K_m(\omega a_d)} \quad (A.7)$$

$$c_m = \left\{ \begin{array}{ll}
\frac{m}{2m + 2(n-1)\pi b/h} & n = 1 \\
\frac{1}{I_m((n-1)\pi b/h)} & n \geq 2
\end{array} \right. \quad (A.8)$$

To compute $F_{\omega}(b_0)$, for each $m$, define vectors $q^m \in \mathbb{R}^{N_n}$ and $g^m \in \mathbb{R}^{N_n}$ element-by-element as

$$q^m = \left[ \begin{array}{c}
\frac{1}{2} \sin^2(-2\omega d) & 
\frac{1}{4\omega d} l_m(-jkb) \\
0 & -jkb
\end{array} \right] : a = 1 \quad (A.9)$$

$$g^m = m + a_m b I_m((n-1)\pi b/h) \quad (A.10)$$

then, define the vectors $p^m \in \mathbb{R}^{N_n}$, $m \in \{0,1\}$ as

$$p^m = \left[ \begin{array}{c}
\frac{\sin(\omega a_d)}{d} \text{diag}(\lambda^m) Q - \text{diag}(w_m) \\
\text{diag}(g^m) - \frac{\sin(\omega a_d)}{d} \text{diag}(\lambda^m) Q \end{array} \right] q^m \quad (A.11)$$

The nonzero elements of the transfer function matrix $F_{\omega}(b_0)$ are then

$$F_{a1}(\omega) = 2\pi \rho \omega^2 d \lambda_{1}^2 \left( q^1 + p^1 \right) \quad (A.12)$$

$$F_{a3}(\omega) = 2\pi \rho \omega^2 d \lambda_{3}^2 Q \left( q^3 + p^3 \right) \quad (A.13)$$

$$F_{a5}(\omega) = -2\pi \rho \omega^2 d \lambda_{5}^2 Q \left( q^5 + p^5 \right) \quad (A.14)$$

To compute $M_{\omega}(\omega)$ and $C_{\omega}(\omega)$, let $Z_{b_0}(\omega)$ be the amplitude (in complex phasor notation) of the hydrodynamic force due to added mass and damping, exerted on degree of freedom $i \in \{1...6\}$ due to the unit-displacement oscillation of degree of freedom $k \in \{1...6\}$. Then from this we can find the corresponding $(i,k)$ components of $C_{\omega}$ and $M_{\omega}$ uniquely, from $Z_{b_0}(\omega) = \{-a^2 M_{\omega}(\omega) + j\omega C_{\omega}(\omega)\}_{ik}$. For the cylinder under consideration, the nonzero values of $Z_{b_0}$ are computed as follows. First compute vectors $q^m \in \mathbb{R}^{N_n}$ and $g^m \in \mathbb{R}^{N_n}$ element-by-element as

$$q_{1n} = 0 \quad (A.15)$$

$$g_{2\omega} = \left\{ \begin{array}{ll}
-hb & n = 1 \\
\frac{1}{4d^2} l_1((n-1)\pi b/h) & n \geq 2
\end{array} \right. \quad (A.16)$$

$$q_{1\omega} = \left\{ \begin{array}{ll}
\frac{h^2 b}{12d^2} - \frac{16d^2}{bh} & n = 1 \\
\frac{1}{2d^2} l_1((n-1)\pi b/h) & n \geq 2
\end{array} \right. \quad (A.17)$$

$$g_{1\omega} = -\frac{b_{a_d}}{2d^2 a_d} \left[\sin(\omega a_d) - \sin(\omega a_1)\right] \quad (A.18)$$

$$g_{2\omega} = \frac{b^2 a_d}{2d^2 a_d} \sin(\omega a_1) \quad (A.19)$$

$$g_{1\omega} = -\frac{b_{a_d}}{2d^2 a_d} \left[\cos(\omega a_d) - 2\cos(\omega a_1) + \sin(\omega a_d)(\omega a_d - \omega a_1)\right] \quad (A.20)$$
Then, for \( m \in \{0, 1\} \) and for each degree of freedom \( k \), find vectors \( p_k^m \in \mathbb{R}^{n_c} \) as
\[
p_k^m = \left[ \frac{1}{d} Q^T \text{diag}(c^m) Q - \text{diag}(w^m) \right]^{-1} \left[ g_k^m - \frac{h}{d} Q^T \text{diag}(c^m)q_k^m \right]
\] (A.21)

In terms of these, compute the nonzero \( Z_{ik} \) components as
\[
Z_{11} = Z_{22} = 2\pi \rho \omega^2 d \bar{T}_1 \bar{T}_1^T
\] (A.22)
\[
Z_{15} = Z_{51} = -Z_{42} = 2\pi \rho \omega^2 d \bar{T}_1 \bar{T}_1^T
\] (A.23)
\[
Z_{33} = 2\pi \rho \omega^2 d \left[ \frac{1}{16\pi a d^2} (4h^2b^2 - b^4) + \lambda_1^T \left( Qp_k^0 + q_k^0 \right) \right]
\] (A.24)
\[
Z_{55} = Z_{44} = -2\pi \rho \omega^2 d \left[ \frac{1}{96\pi d^2} (6b^4h^2 + b^6) + \lambda_1^T \left( Qp_k^0 + q_k^0 \right) \right]
\] (A.25)

**Appendix B. Steps for subspace-based finite-dimensional approximation**

Let \( G(s) \) be a generic transfer function of dimension \( p \times m \), which we wish to estimate as a finite-dimensional state space. Denote this approximate transfer function as \( \tilde{G}(s) \). The procedure is parametrized by four variables \( (T, M, q, r) \). To generate data in this paper, these parameters were uniformly set to \( (0.5, 257, 257, 257) \). However, the algorithm can be made more efficient (at the expense of accuracy) by reduction of \( q \) and \( r \). These tradeoffs were not explored here.

1. To begin, we must evaluate \( G(\omega) \) (including the solution to the frequency-dependent hydrodynamic coefficients) at discrete values of \( \omega \). Here, it is convenient to use a very specific array of frequency values. To define these, we first define the set \( \Omega_{k} = \pi k/M, k \in \{0, \ldots, M\} \) for some \( M \in \mathbb{Z} \). Then, we define the evaluation frequencies \( \{\omega_0, \omega_1, \ldots, \omega_{M-1}\} \) through a bilinear transformation, as
\[
\omega_k := \frac{2}{T} \tan \left( \frac{\Omega_k}{2} \right)
\] (B.1)

For convenience, define:
\[
G_k \equiv \begin{cases} G(\omega_k) & k \in \{0, \ldots, M\} \\ G^\ast(\omega_{M-k}) & k \in \{M+1, \ldots, 2M-1\} \end{cases}
\] (B.2)

2. Compute the inverse discrete Fourier transform (DFT) of \( G_k \), \( k \in \{0, \ldots, 2M-1\} \) as
\[
\tilde{g}_\ell = \frac{1}{2M} \sum_{k=0}^{2M-1} G_k e^{j\Omega_k \ell}
\] (B.3)
The resultant coefficients \( \tilde{g}_\ell, \ell \in \{0, \ldots, 2M-1\} \) approximate the impulse response of the discrete-time transfer function \( \tilde{G}(z) \).

3. For two coefficients \( q, r \) such that \( q + r \leq 2M \), construct the Hankel matrix
\[
H_{qr} = \begin{bmatrix} \tilde{g}_0 & \cdots & \tilde{g}_q \\ \vdots & \ddots & \vdots \\ \tilde{g}_r & \cdots & \tilde{g}_{q+r-1} \end{bmatrix}
\] (B.4)
and perform a singular value decomposition as
\[
H_{qr} = [U_q U_r] \begin{bmatrix} \Sigma_q & 0 \\ 0 & \Sigma_r \end{bmatrix} \begin{bmatrix} V_q^\ast \\ V_r^\ast \end{bmatrix}
\] (B.5)
where both \( \Sigma_q \) and \( \Sigma_r \) are each diagonal and contain the singular values in decreasing order. The partitioning of the matrices above is determined such that the singular values in \( \Sigma_q \) are zero to numerical precision, while those in \( \Sigma_r \) are deemed significant. Let the number of significant singular values be \( n_G \) (chosen as 12 for \( G_9 \) and 10 for \( G_9 \) in this study).

4. Construct an approximate finite-dimensional discrete-time system as
\[
\tilde{G}(z) = D + C [zI - A]^{-1} B.
\] (B.6)
where
\[
A = \begin{bmatrix} (q-1)p & \text{rows of } U_{q+1}^\ast \\ (q-1)p & \text{rows of } U_1 \end{bmatrix}
\] (B.7)
\[
C = \begin{bmatrix} \text{first } p & \text{rows of } U_1 \end{bmatrix}
\] (B.8)
\[
B = \left[ I - A^{2M} \right] \Sigma_1 \left[ \begin{array}{c} \text{first } m & \text{rows of } V_1 \end{array} \right]^T
\] (B.9)
\[
D = \Sigma_0 - C A^{2M-1} \Sigma_1 \left[ \begin{array}{c} \text{first } m & \text{rows of } V_1 \end{array} \right]^T
\] (B.10)
and where \( (\cdot)^\ast \) denotes the left pseudoinverse. This finite-dimensional system approximates the discrete-time transfer function \( \tilde{G}(z) \). The computed \( A \) matrix is then checked to verify that it is discrete-time asymptotically stable (i.e., that its poles are inside the unit disk). Any poles outside the unit disk are radially reflected inside the disk, and then \( B \) and \( D \) are recalculated.

5. The matrix parameters \( \{A, B, C, D\} \) for the approximate continuous-time system
\[
G'(s) = \tilde{D} + C [sI - A]^{-1} B
\] (B.11)
is then found via the standard bilinear (i.e., Tustin) transformation using sample time \( T \), which can (for example) be evaluated using the \texttt{d2c} command in Matlab. Although \( D \) may be nonzero in the finite-dimensional approximation, it is usually small, and is discarded.

**Appendix C. WSPR adjustment algorithm for finite-dimensional \( G(s) \)**

For finite-dimensional and strictly-proper transfer functions, the WSPR property holds if and only if there exists a matrix \( W_1 = W_1^\ast > 0 \) such that \( B_h = W_1 C_h^\ast \) and the following matrix inequality holds:
\[
A_h W_1 + W_1 A_h^\ast < 0
\] (C.1)

Suppose that we only adjust \( B_h \) to bring about the WSPR condition; i.e., \( A_h = A_h, C_h = C_h, B_h = B_h \). Then the error system \( \Delta(s) = G(s) - G_h(s) \) is
\[
\Delta(s) = C_h s I - A_h \right)^{-1} [B_h - B_h 0].
\] (C.2)

It can be shown that \( \|\Delta\|_{\infty} < \gamma \) for the above parametrization if and only if there exists a matrix \( W_2 = W_2^\ast > 0 \) such that
\[
\begin{bmatrix} A_h W_2 + W_2 A_h^\ast & B_h - B_h^\ast W_2 C_h^\ast \\ B_h^\ast - B_h & -\gamma 1 & 0 \\ C_h W_2 & 0 & -\gamma 1 \end{bmatrix} < 0
\] (C.3)

Thus formulated, we seek positive-definite matrices \( \{W_1, W_2\} \) such that with \( W_1 C_h^\ast = B_h \), the quantity \( \gamma \) is minimized, subject to linear matrix inequality (LMI) constraints (C.1) and (C.3). Substituting \( W_1 C_h^\ast = B_h \) into (C.3), we see that this matrix inequality is linear in the unknowns \((\gamma, W_1, W_2)\). As such, determining the \( W_1 \) that minimizes \( \gamma \) reduces to a convex, linear semidefinite program:
\[
W_1 = \begin{cases} \text{minimize:} & \gamma \\ \text{over:} & \gamma, W_1, W_2 \\ \text{constraints:} & W_1 > 0, W_2 > 0, (C.1) \text{ and } (C.3) \end{cases}
\] (C.3)

Such semidefinite programs can be solved extremely efficiently using many commercially-available software packages, such as the linear matrix inequality solver included in Matlab’s robust control toolbox.
With a solution found as above for $W_1$, the adjusted $B_h$ matrix is
\[ B_h = W_1 C_h. \]

References


